Contraction Analysis: a Geometric Viewpoint

Friday Seminar at LTH, Department of Automatic Control

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Work directed by Prof. Antoine Chaillet* and Prof. Guangren Duan†

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About me

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Dissipative methods

(with Prof. R. Ortega and G. Duan)

Past researches

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Contraction analysis
(with Prof. A. Chaillet)

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Contraction analysis
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What is contraction?

Nonlinear system

$$\dot{x} = f(x), \quad x \in C \subset \mathcal{M} \tag{1}$$

Contraction on C

All solutions in C converge toward each other.

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Formal definition:

1. Asymptotic contraction or (IAS):

$$d(X(t,x_1), X(t,x_2)) \to 0, \quad \forall t \geq 0, x_1, x_2 \in C$$

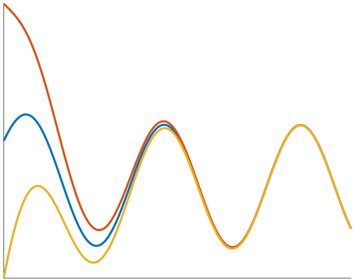
2. Exponential contraction or (IES)

$$d(X(t,x_1), X(t,x_2)) \leq Ke^{-\lambda t} d(x_1, x_2), \quad \forall t \geq 0, x_1, x_2 \in C$$

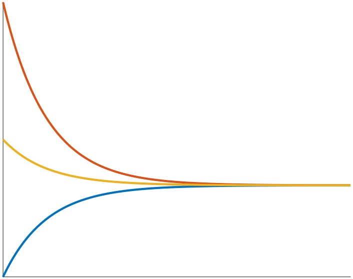
A remark on terminology

incremental asymptotic stability (IAS)	incremental exponential stability (IES)
$\Updownarrow$	$\Updownarrow$
asymptotic contraction	exponential contraction

Contraction vs. Stability



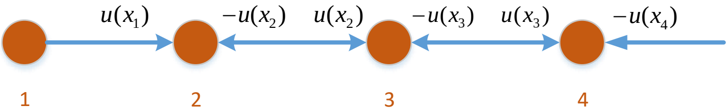
Contraction: target solution may be unknown



Stability: equilibrium as target solution

Why do we need contraction?

Synchronization as contraction



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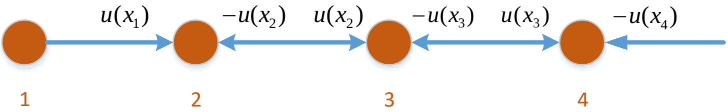

$$\Sigma : \begin{cases} \dot{x}_1 = f(x_1) \\ \dot{x}_2 = f(x_2) - u(x_2) + u(x_1) \\ \dot{x}_3 = f(x_3) - u(x_3) + u(x_2) \\ \dot{x}_4 = f(x_4) - u(x_4) + u(x_3) \end{cases}$$

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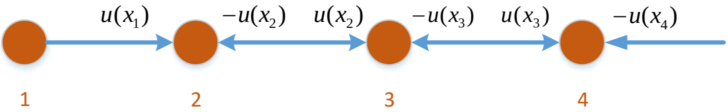

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$\dot{x} = f(x) - u(x)$ IES $\Rightarrow$ Σ synchronizes

Contraction based observer

$$\Sigma : \begin{cases} \dot{x} = f(x), \ x \in \mathbb{R}^n \\ y = h(x), \ y \in \mathbb{R}^m \end{cases}, \qquad y\text{: the measurement.}$$

Contraction-based observer:

$$\exists g \in C^1, \text{ s.t. } f(x) = g(x, h(x))$$

and:

$$\dot{x} = g(x, y) \qquad \text{IES (uniform in } y\text{)}$$

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$$\Downarrow$$

Observer: $\dot{\hat{x}} = g(\hat{x}, y)$.

Basic question

How to analyze incremental stability? (= contraction analysis)

Set stability $\Rightarrow$ incremental stability

Euclidean space,

$$\Sigma : \begin{cases} \dot{x} = f(x) \\ \dot{z} = f(z) \end{cases}, \quad x, z \in \mathbb{R}^n$$

Diagonal set: $A = \{(x, z) \in \mathbb{R}^{2n} | x = z\}$.

A exponentially stable $\quad \Rightarrow \quad \dot{x} = f(x)$ IES.

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LF-based criteria ⇒ incremental stability. [1]

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Remarks

(1). Construction of a set LF *relies on distance* $|x - y|$, *sometimes difficult to calculate*, e.g. systems on manifolds.
(2). System Σ contains two copies of the *same* system: redundancy.

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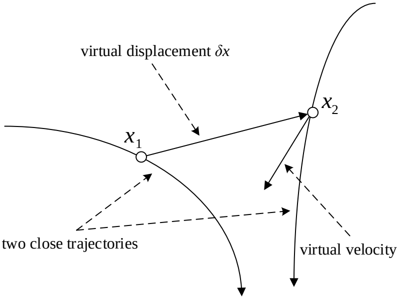
Differential contraction analysis



$x_1(t)$ and $x_2(t)$ sufficiently close

$$\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2) \approx \frac{\partial f(x)}{\partial x}(x_1 - x_2),$$

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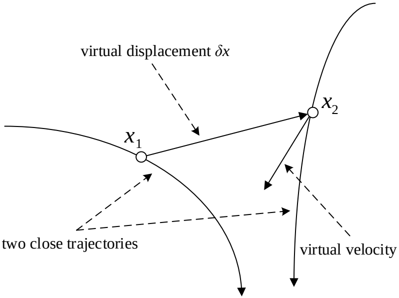

$x_1(t)$ and $x_2(t)$ sufficiently close

$$\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$$
$$\approx \frac{\partial f(x)}{\partial x}(x_1 - x_2),$$

The "error dynamics" $x_1 - x_2$ is characterized by $\frac{\partial f(x)}{\partial x}$.

W. Lohmiller and J.J. Slotine[2]:

virtual dynamics: $$\delta \dot{x} = \frac{\partial f(x)}{\partial x} \delta x.$$

[2] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998

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In particular: $\exists P > 0,\ c > 0$ s.t. $$P\frac{\partial f(x)}{\partial x} + \frac{\partial^T f(x)}{\partial x}P \leq -cI, \quad \forall x \in \mathbb{R}^n$$

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Remarks

(1) Advantage: no need of system augmentation or distance.
(2) The mathematical meaning behind this idea needs to be further justified: approximation, infinitesimal analysis, virtual dynamics.

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Finsler-Lyapunov functions

F. Forni and R. Sepulchre:[3]

Theorem

If ∃ a "Finsler-Lyapunov function" $V : TM \to \mathbb{R}_+$, satisfying, for all $(x, \delta x) \in TM$,

$$c_1|\delta x|^p \leq V(x,\delta x) \leq c_2|\delta x|^p, \quad (2)$$

$$\frac{\partial V(x,\delta x)}{\partial x} f(x) + \frac{\partial V(x,\delta x)}{\partial \delta x}\frac{\partial f(x)}{\partial x}\delta x \leq -\alpha(V(x,\delta x)). \quad (3)$$

for some $c_1, c_2 > 0, p \geq 1$, then the system is

- *incrementally asymptotically stable (IAS), if α is class $\mathcal{K}$.*
- *incrementally exponentially stable (IES), if $\alpha(s) = \lambda s, \lambda > 0$.*

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Remarks

(1). Introduce new objects to study contraction.
(2). Provide more rigorous interpretation to differential contraction analysis.
(3). Limitations: local results; geometric meaning not quite clear; sufficient.

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Why geometric contraction analysis

Growing needs of intrinsic methods (intrinsic observers in particular) for systems on manifolds:

- Rigid body dynamics: $SO(3)$, $SE(3)$.
- Mechanical systems: $\mathbb{R}^m \times (\mathbb{S}^1)^k$.
- Quantum systems: $SU(3)$ etc.

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Limitation seen from the literature:

- Results mainly focused on Euclidean space.
- Local results are sometimes cumbersome for theoretical analysis.

Objectives

1. Gain the geometric understandings of contraction.

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2. Develop framework for intrinsic contraction analysis on manifolds.

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1. Gain the geometric understandings of contraction.
2. Develop framework for intrinsic contraction analysis on manifolds.
3. Solve some challenging problems using intrinsic contraction analysis.

Ready? Go!

Stability analysis of trajectories on manifolds

- System: $\dot{x} = f(t, x, u)$
- A trajectory: $q(\cdot)$, s.t. $\dot{q}(t) = f(t, q(t), u_*(t))$, u_* an input.

Stability analysis of trajectories on manifolds

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Task

Analyze the stability of q.

Stability analysis of trajectories on manifolds

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Analyze the stability of q.

If $M = \mathbb{R}^n$, define $e = x - q(t) \quad \Rightarrow \quad$ error dynamics:

$$\dot{e} = f(t, e + q(t), u_*(t)) - \dot{q}(t)$$

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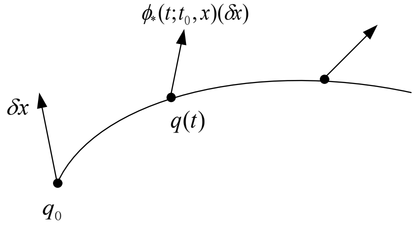
On manifold: $x - q(t)$ no longer makes sense; no standard error; depends heavily on the choice of error dynamics, often non-trivial!

Lifting technique: linearization on manifolds



Step 1: let $\delta x(t) = \phi_*(t;t_0,x)(\delta x) = \mathrm{Lie}(\delta x)(t;t_0)$.
Step 2: $\Gamma(t) = (q(t), \delta x(t))$: curve in TM.
Step 3: The complete lift of $f(t,x)$ along $q(\cdot)$:

$$\tilde{f}(t, q(t), \delta x(t)) = \frac{d\Gamma(t)}{dt} \in T_{(q(t),\delta x(t))}TM, \quad \forall \delta x \in T_{q_0}M$$

complete lift system $\Longleftrightarrow$ linearization of error dynamics

The vector field $\tilde{f}$ defines a system:

$$\dot{v} = \tilde{f}(t, q(t), v), \quad v \in T_{q(t)}M \tag{4}$$

- System is fibre-wise linear!
- $v_1, v_2 \in q^*TM$ solve (4) $\Rightarrow$ so is $\alpha_1 v_1 + \alpha_2 v_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

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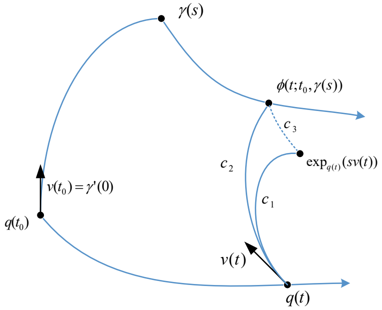
Theorem (D. Wu *et al.* 2019)

Assume that (4) is the complete lift along q of the system

$$\dot{x} = f(t, x). \tag{5}$$

- *(q LES) $\Rightarrow$ (CL system ES).*
- *Periodic system, q bounded, then (CL system ES) $\Rightarrow$ (q LES)*

Necessity: (q LES) $\Rightarrow$ (CL system ES)



Show:

$$|v(t)| \leq ke^{-\lambda(t-t_0)}|v(t_0)|$$

- γ: geodesic starting at $q(t_0)$.
- c_1: geodesic of length $s|v(t)|$.
- c_2: flow of $\gamma : [0, s] \to M$.
- c_3: geodesic $\phi(t; t_0, \gamma(s))$ to $\exp_{q(t)}(sv(t))$.

$$s|v(t)| = \ell(c_1) = d(q(t), \exp_{q(t)}(sv(t))) \leq \ell(c_2) + \ell(c_3)$$

$$\ell(c_2) \leq ks|v(t_0)|e^{-\lambda(t-t_0)}$$

It suffices to show

$$\frac{\ell(c_3)}{s} = \frac{d(\phi(t; t_0, \gamma(s)), \exp_{q(t)}(sv(t)))}{s} \to 0$$

as $s \to 0+$.

Sufficiency: (CL syst. along q ES) $\Rightarrow$ q LES.

CL system ES $+$ fibre-wise linear

$$\Downarrow$$

$$\exists V : \mathbb{R}_+ \times q^*TM \to \mathbb{R}_+ \quad \text{s.t.} \quad \begin{cases} c_1|v|^2 \leq V(t,v) \leq c_2|v|^2 \\ \mathcal{L}_{\tilde{f}} V(t,v) \leq -c_3|v|^2. \end{cases} \tag{6}$$

Extend the V to D $\leftarrow$ a bounded open neighborhood containing $q(\cdot)$

$$\Downarrow$$

The system is IES on D

$$\Downarrow$$

$q(\cdot)$ LES

Corollaries

Corollary

(1) If $q(\cdot)$ is a trajectory of $\dot{x} = f(x)$, and $\exists k > 0$ such that

$$\langle \nabla_v f(x), v\rangle|_{x=q(t)} \leq -k\langle v, v\rangle, \tag{7}$$

$\forall v(t) \in T_{q(t)}M,\ t \geq 0$, *then $q(\cdot)$ is LES.*
(2) For autonomous system, $\nexists$ nontrivial bounded LES trajectory.

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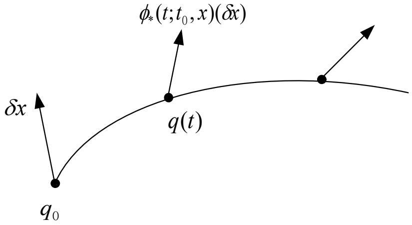
(2) For autonomous system, $\nexists$ nontrivial bounded LES trajectory.

(1): No need to calculate complete lift.

(2): Limit cycle of autonomous system cannot be LES.

Lift along q



$$\Gamma(s) = (q(s), \phi_*(s;t,x)(\delta x))$$

$$\tilde{f}(t, q(t), \delta x) = \left.\frac{d\Gamma(s)}{ds}\right|_{s=t} \in T_{(q(t),\delta x)}TM$$

$$\Downarrow$$

$$\dot{v} = \tilde{f}(t, v), \; v(t) \in T_{q(t)}M \quad (8)$$

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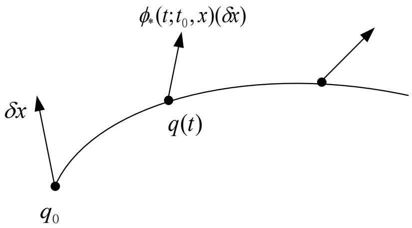

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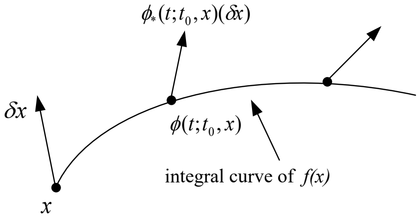
Lift to TM



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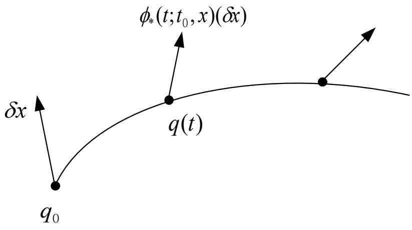
Lift along q



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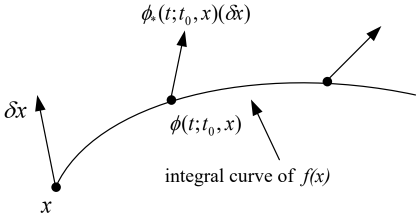
Lift to TM



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$$\tilde{f}(t, x, \delta x) = \left.\frac{d\Gamma(s)}{ds}\right|_{s=t} \in T_{(x,\delta x)}TM$$

$$\Downarrow$$

$$\dot{v} = \tilde{f}(t,v),\ v(t) \in TM \quad (9)$$

Theorem (D. Wu *et al.* 2020)

Consider the system $\dot{x} = f(t,x)$, *a* $\mathcal{K}$ *function* α, *and the CL system*

$$\dot{v} = \tilde{f}(t,v), \quad v \in TM$$

Let V *be a candidate Finsler-Lyapunov function, i.e.,* $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that* $\forall (t,v) \in \mathbb{R}_+ \times TM$:

$$\alpha_1(|\delta x|) \leq V(t,x,\delta x) \leq \alpha_2(|\delta x|) \tag{10}$$

$$\mathcal{L}_{\tilde{f}} V(t,v) \leq -\alpha(V(t,v)) \tag{11}$$

then the system is

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$$\mathcal{L}_{\tilde{f}} V(t,v) \leq -\alpha(V(t,v)) \tag{11}$$

then the system is

- *incrementally asymptotically stable (IAS) if* α *is* $\mathcal{K}$;
- *incrementally exponentially stable (IES) if* $\alpha(s) = \lambda s$, $\lambda > 0$.

Remarks

(1). The theorem is intrinsic
(2). Recover F. Forni and R. Sepulchre's results.
(3). α_1 and α_2 only $\mathcal{K}_\infty$, even for IES.

Theorem (D. Wu *et al.* 2020)

Consider the system $\dot{x} = f(t,x)$, *a* $\mathcal{K}$ *function* α, *and the CL system*

$$\dot{v} = \tilde{f}(t,v), \quad v \in TM$$

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Recall[4]

Theorem

If ∃ a "Finsler-Lyapunov function" $V : TM \to \mathbb{R}_+$, satisfying

$$c_1|\delta x|^p \leq V(x,\delta x) \leq c_2|\delta x|^p, \ \forall (x,\delta x) \in TM, \tag{12}$$

$$\frac{\partial V(x,\delta x)}{\partial x} f(x) + \frac{\partial V(x,\delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} \delta x \leq -\alpha(V(x,\delta x)). \tag{13}$$

for $\forall (x,\delta x) \in TM$ *and some* $c_1, c_2 > 0, p \geq 1$*, then the system is*

- ***incrementally asymptotically stable*** *(IAS), if* α *is class* $\mathcal{K}$*.*
- ***incrementally exponentially stable*** *(IES), if* $\alpha(s) = \lambda s, \lambda > 0$*.*

[4] F. Forni and R. Sepulchre, "A differential lyapunov framework for contraction analysis," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 614–628, 2014.

In local coordinate

Lemma

Let $\{x, v\}$ *be the local coordinate of* TM*, and* TTM *is spanned by* $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v_i}\right\}$*.*
Then $\tilde{f}(t, v)$ *is expressed as*

$$\tilde{f} = \begin{bmatrix} f(t, \pi(v)) \\ \frac{\partial f}{\partial x}(t, \pi(v))v \end{bmatrix}.$$

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The CL of $\dot{x} = f(t, x)$ now reads

$$\begin{cases} \dot{x} = f(t, x) \\ \delta\dot{x} = \dfrac{\partial f(t, x)}{\partial x}\delta x. \end{cases} \tag{14}$$

And $\mathcal{L}_{\tilde{f}}V$:

$$\mathcal{L}_{\tilde{f}}V(x, \delta x) = \frac{\partial V(x, \delta x)}{\partial x}f(t, x) + \frac{\partial V(x, \delta x)}{\partial \delta x}\frac{\partial f(t, x)}{\partial x}\delta x \tag{15}$$

Converse Theorem

Sufficient condition for contraction obtained. Is it also necessary?

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Answer: YES!

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Is Finsler-Lyapunov function the "right" measure of contraction?

Theorem (D. Wu *et al.* 2020)

Consider $\dot{x} = f(t,x),\ x \in M,\ f \in C^1$, $|P_p^q f(t,p) - f(t,q)| \le L d(p,q)$ *for all* $p, q \in M$ *and some constant* $L > 0$. *Then the system is IES* **iff** *there exists a Finsler-Lyapunov function and* $c_1, c_2, k > 0$ *s.t.*

1. *There exist constants* c_1, c_2 *such that*

$$c_1|v|^2 \le V(t,v) \le c_2|v|^2, \quad \forall (t,v) \in \mathbb{R}_+ \times TM$$

2. *There exists constant* $k > 0$ *such that*

$$\mathcal{L}_{\tilde{f}} V(t,v) \le -kV(t,v), \quad \forall (t,v) \in \mathbb{R}_+ \times TM \tag{16}$$

A "near perfect" correspondence

Contraction	**Lyapunov Stability**
State space: TM	State space: M
Finsler-Lyapunov function	Lyapunov function
$\alpha_1(\|\delta x\|) \leq V(t,x,\delta x) \leq \alpha_2(\|\delta x\|)$	$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$
$\mathcal{L}_{\tilde{f}} V(t,x,\delta x) \leq -\alpha_3(\|\delta x\|)$	$\mathcal{L}_f V(t,x) \leq -\alpha_3(\|x\|)$

New insights obtained!

But wait, what can we do with this?

Krasovskii Theorem and contraction

Classical Krasovskii Thm

Syst $\dot{x} = f(x)$, if $\exists\ P > 0, k > 0$, s.t.

$$P\frac{\partial f}{\partial x} + \frac{\partial^T f}{\partial x}P \leq -kI \qquad (17)$$

(Demidovich condition), $f(0) = 0$, then a Lyapunov func can be constructed[a]:

$$V(x) = f^T(x)Pf(x), \text{ s.t. } \dot{V} \leq -kV,$$

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Contraction viewpoint

If (17) holds, a FLF can be constructed!

$$W(x, \delta x) = \delta x^T P \delta x, \qquad (18)$$

$$\mathcal{L}_{\tilde{f}} W(x, \delta x) \leq -kW(x, \delta x) \qquad (19)$$

(18) + (19) $\Rightarrow$ IES $\Rightarrow$ 0 ES

What is the LF?

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Question

(IES) + ($\exists$ equilibrium) $\Longrightarrow$ How to construct LF?

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A generalized Krasovskii Theorem

Theorem (D. Wu *et al* 2020)

If the system $\dot{x} = f(x)$ IES and $f(0) = 0$, with a FLF: $V(x, \delta x)$, then

- *The system is ES;*
- *If $\exists$ C^1 vector field h, with $h(x) = 0$ iff $x = 0$ and that $[f, h] = 0$, (in particular $h = f$)*

$$W(x) = V(x, h(x))$$

is a Lyapunov function for the system.

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is a Lyapunov function for the system.

Remarks

- h not necessarily f;
- holds on manifolds;
- need not be quadratic.
- connection to switch system: commutativity implies GES under arbitrary switching!

A tricky proof in Euclidean space

Theorem (D. Wu *et al* 2020)

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$$W(x) = V(x, h(x))$$

is a Lyapunov function for the system.

Proof in Euclidean space

$$\begin{aligned}
\dot{W} &= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial \delta x} \frac{\partial h}{\partial x} f(x) \\
&= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial \delta x} \frac{\partial f}{\partial x} h(x) \qquad \left(\text{since } \frac{\partial h}{\partial x} f = \frac{\partial f}{\partial x} h\right) \\
&\leq -kV(x, h(x)) \qquad \left(\text{since } \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial \delta x} \frac{\partial f}{\partial x} \delta x \leq -kV\right) \\
&= -kW
\end{aligned}$$

Local property (D. Wu *et al.* 2021)

Recall that to guarantee IES, the following needs to be hold for all $(x, \delta x) \in TM$,

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial \delta x} \frac{\partial f}{\partial x} \delta x \leq -kV, \tag{20}$$

This can be relaxed to

$$\forall |\delta x| < c, (x, \delta x) \in TM$$

where c is any positive constant.

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Local implies global!

Remarks

(1). Not trivial from (20) which is not linear in δx!
(2). Key to proof: reparametrize geodesics.

IES and LES of trajectories

LES

complete lift along trajectory

IES

complete lift everywhere

IES and LES of trajectories

LES	IES
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Similar tool should imply some kind of equivalence!

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Theorem (D. Wu *et al.* 2020)

Consider the system

$$\dot{x} = f(t, x) \tag{21}$$

which is autonomous or periodic, q is a bounded solution. Then q is LES if and only if $\exists$ an open invariant neighborhood of q, on which the system is IES.

LES of trajectories $\sim$ IES on a region

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LES of trajectories $\sim$ IES on a region

Particular case: x_* is LES iff $\exists U \ni x_*$ s.t. the system is IES on U. [5]

[5] F. Forni, A. Mauroy, and R. Sepulchre, "Differential positivity characterizes one- dimensional normally hyperbolic attractors," arXiv preprint arXiv:1511.06996, 2015

Volume shrinking

Demidovich condition

$$P\frac{\partial f}{\partial x} + \frac{\partial^T f}{\partial x}P \leq -cI.$$

on Riemannian manifolds

$$\langle \nabla_v f, v\rangle \leq -c|v|^2, \tag{22}$$

Theorem (D. Wu *et al.* 2021)

If the system $\dot{x} = f(t,x)$ satisifies (22), then for any open set D with C^1 boundary, vol(D) decreases exponentially.

Proof. (valid on Riemannian manifold).

- By transport formula, $\frac{d}{dt}\text{vol}(D_t) = \int_{D_t}(\text{div} f)\text{vol}$
- $\text{div} f = \text{tr}(\nabla f)$

□

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Proof in Euclidean space

Proof.

In Euclidien space

$$\frac{d}{dt}\,\text{vol}(D_t) = \int_{D_t} \text{div}\, f dx \tag{23}$$

$$= \int_{D_t} \text{tr}\left(\frac{\partial f}{\partial x}\right) dx \le \int_{D_t} -\frac{nc}{2a} dx \tag{24}$$

$$= -\frac{nc}{2a}\,\text{vol}(D_t) \tag{25}$$

(24) is true since

$$P\frac{\partial f}{\partial x} + \frac{\partial^T f}{\partial x} P \le -\frac{c}{a} P, \tag{26}$$

implies $\text{Re}(\sigma(\partial f/\partial x)) \le -c/(2a)$ (weaker than contraction!) □

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Extremum seeking on Riemannian manifolds

Assume the syst $\dot{x} = f(x)$ on M satisfies

$$\langle \nabla_v f, v \rangle \leq -c|v|^2, \quad \forall v \in TU \tag{27}$$

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Extremum seeking on Riemannian manifolds

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Find x_* numerically.

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Algorithm: $x_{k+1} = \exp_{x_k} \alpha f(x_k)$, $\alpha > 0$, exp: Riemannian exponential map.

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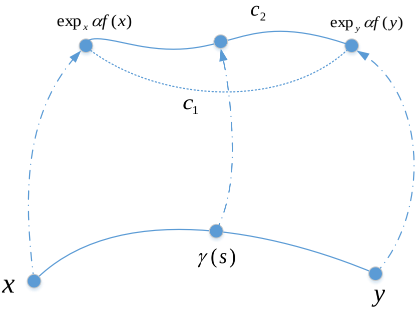
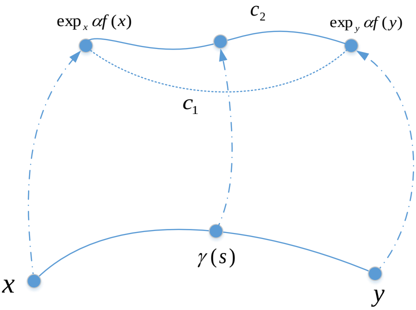
Task 2

Find the optimal α s.t. the algorithm converges at the fastest rate.



- show $x \mapsto \exp_x \alpha f(x)$ is Banach contraction.
- estimate $d(\exp_x(\alpha f(x)), \exp_y(\alpha f(y)))$.
- γ: geod. joining x to y, length ℓ.
- c_1: geod. $\exp_x(\alpha f(x))$ to $\exp_y(\alpha f(y))$
- c_2: image of γ.



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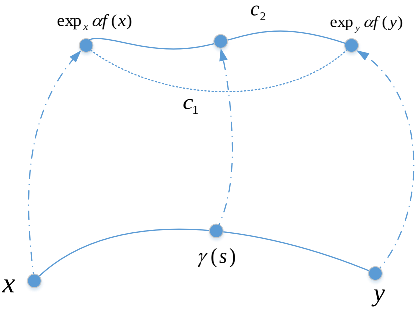
$$d(\exp_x(\alpha f(x)), \exp_y(\alpha f(y))) \leq \int_0^{\ell} \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s))) \right| ds \tag{28}$$



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$\frac{d}{ds} \exp_{\gamma(s)}(\alpha f(\gamma(s)))$ is the Jacobi field!

i.e., the solution to

$$J_s''(r) + R(\varphi_s'(r), J_s(r))\varphi_s'(r) = 0$$
$$J_s(0) = \gamma'(s), \; J_s'(0) = \nabla_{\gamma'} f(\gamma(s))$$

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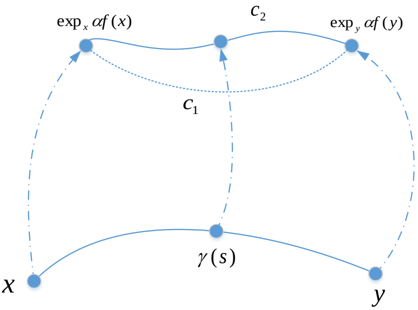

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Curvature tensor comes into play!

Estimation of the Jacobi field

$$\int_0^\ell \langle J_s(\alpha), J_s(\alpha)\rangle\, ds = 2\int_0^\ell \int_0^\alpha \left\langle J_s'(r), J_s(r)\right\rangle drds + \int_0^\ell |J_s(0)|^2 ds$$

$$= 2L + \int_0^\ell |\gamma'(s)|^2 ds = 2L + \ell$$

$$L = \int_0^\ell \int_0^\alpha \left\langle J_s'(r), J_s(r)\right\rangle drds$$

$$= \int_0^\ell \int_0^\alpha \left(\int_0^r \frac{d}{dt}\left\langle J_s'(t), J_s(t)\right\rangle dt + \left\langle J_s'(0), J_s(0)\right\rangle\right) drds$$

$$= \int_0^\ell \int_0^\alpha \left(\int_0^r \left\langle J_s''(t), J_s(t)\right\rangle + \left\langle J_s'(t), J_s'(t)\right\rangle dt + \left\langle J_s'(0), J_s(0)\right\rangle\right) drds$$

$$= \int_0^\ell \int_0^\alpha \int_0^r (\left\langle -2R(\varphi_s'(t), J_s(t))\varphi_s', J_s(t)\right\rangle + U(t,s) + \left\langle J_s'(0), J_s(0)\right\rangle)dtdrds$$

$$\le \int_0^\ell \int_0^\alpha \left(\int_0^r U(0,s)dt\right) drds + \int_0^\ell \int_0^\alpha \left\langle J_s'(0), J_s(0)\right\rangle drds$$

$$\le \frac{1}{2}\alpha^2 \int_0^\ell U(0,s)ds - c\int_0^\ell \int_0^\alpha drds \tag{29}$$

Calculation results

For *non-negative constant curvature* K manifold

$$
\begin{aligned}
d(\exp_x(\alpha X(x)), \exp_y(\alpha X(y))) &\le \int_0^\ell \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha X(\gamma(s))) \right| ds \\
&\le \sqrt{\ell} \sqrt{\int_0^\ell \langle J_s(\alpha), J_s(\alpha) \rangle \, ds} \\
&\le \ell \sqrt{1 - 2c\alpha + \alpha^2 (1+K) L^2} \\
&= \sqrt{1 - 2c\alpha + \alpha^2 (1+K) L^2} \, d(x,y)
\end{aligned} \tag{30}
$$

- c: the contraction rate of IES
- K: the curvature
- L: Lipschitz constant on Riemannan manifolds

Calculation results

For *non-negative constant curvature* K manifold

$$
\begin{aligned}
d(\exp_x(\alpha X(x)), \exp_y(\alpha X(y))) &\le \int_0^\ell \left| \frac{d}{ds} \exp_{\gamma(s)}(\alpha X(\gamma(s))) \right| ds \\
&\le \sqrt{\ell} \sqrt{\int_0^\ell \langle J_s(\alpha), J_s(\alpha) \rangle \, ds} \\
&\le \ell \sqrt{1 - 2c\alpha + \alpha^2 (1+K) L^2} \\
&= \sqrt{1 - 2c\alpha + \alpha^2 (1+K) L^2} d(x,y)
\end{aligned} \tag{30}
$$

- c: the contraction rate of IES
- K: the curvature
- L: Lipschitz constant on Riemannan manifolds

optimal α : $\alpha_* = \dfrac{c}{(1+K)L^2}$, contraction rate$^{-1} = \sqrt{1 - \dfrac{c^2}{(1+K)L^2}}$

Hor d'oeuvre

Newton's 2nd law for free motion:

$$\ddot{q} = 0, \tag{31}$$

Matrix form:

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q \\ v \end{bmatrix} \\ y = q \end{cases} \tag{32}$$

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Standard Luenberger observer:

$$\begin{cases} \dot{\hat{q}} = \hat{v} - \alpha(\hat{q} - q) \\ \dot{\hat{v}} = -\beta(\hat{q} - q) \end{cases}, \quad \alpha, \beta > 0 \tag{33}$$

Moving from Euclidean to Riemannian

System: $\nabla_{\dot{q}}\dot{q} = 0$ or

$$\dot{q} = v,\ \nabla_{\dot{q}} v = 0, \quad q \in M,\ v \in T_q M \tag{34}$$

[6] N. Aghannan and P. Rouchon, "An intrinsic observer for a class of lagrangian systems," IEEE Transactions on Automatic Control, vol. 48, no. 6, pp. 936–945, 2003.

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Speed observer: reconstruct v using q. Consider [6]

$$\begin{cases} \dot{\hat q} = \hat v - \alpha \operatorname{grad} F(\hat q, q) \\ \nabla_{\dot{\hat q}} \hat v = -\beta \operatorname{grad} F(\hat q, q) \end{cases} \tag{35}$$

F: square distance, R: curvature tensor, grad: gradient

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Task

Analyze the convergence of the observer.

[6] N. Aghannan and P. Rouchon, "An intrinsic observer for a class of lagrangian systems," IEEE Transactions on Automatic Control, vol. 48, no. 6, pp. 936–945, 2003.

Contraction analysis in local coordinates

N. Aghannan and P. Rouchon [7]

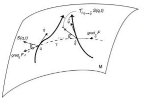


Fig. 1. Parallel transport of the source term $S(q,t)$ from the tangent space at q to the tangent space at $\hat{q}$, along the geodesic γ.

- $\mathcal{T}_{//q\to\hat{q}}$ is the parallel transport from q to $\hat{q}$ along the geodesic between q and $\hat{q}$. It is a linear isometry from the tangent space at q to the tangent space at $\hat{q}$. As for F, this operator is well defined for q and $\hat{q}$ close enough.
- R is the curvature tensor.

In local coordinates (q^i), the observer dynamics reads

$$\dot{\hat{q}}^i = \hat{v}^i - \alpha\{\mathrm{grad}_{\hat{q}} F(\hat{q}, q)\}^i$$
$$\dot{\hat{v}}^i = -\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k + \{\mathcal{T}_{//q\to\hat{q}} S(q,t) - \beta\,\mathrm{grad}_{\hat{q}} F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}} F(\hat{q},q))\hat{v}\}^i.$$

This observer does not depend on the choice of a particular set of coordinates for q: the connection ∇, the function F, the operator $\mathcal{T}_{//q\to\hat{q}}$ and the tensor R are intrinsic objects attached to the Riemannian structure on M.

Fig. 1 illustrates the observer dynamics (2). As the configuration space M has a Riemannian structure, we cannot compare vectors living in tangent spaces at different points on M, as it is usually done in Euclidian spaces. Indeed, we have to take into account the curvature introduced by the metric: the output injection term $\mathcal{T}_{//q\to\hat{q}} S(q,t)$ belongs to the tangent space $T_{\hat{q}}M$, whereas $S(q,t)$ belongs to $T_q M$ and cannot be combined to $\nabla_{\hat{q}}\hat{v} \in T_{\hat{q}}M$. We also replace the often used error term $(\hat{q} - q)$ by $\mathrm{grad}_{\hat{q}} F(\hat{q}, q)$ to deal with the curvature since it gives the direction by which q can be "joined" from $\hat{q}$ by taking the shortest path. The term $-\beta\,\mathrm{grad}_{\hat{q}} F(\hat{q},q)$ can be interpreted as a spring term if we consider its counterpart in the Euclidian case as described in the introduction. The term $R(\hat{v}, \mathrm{grad}_{\hat{q}} F(\hat{q},q))\hat{v}$ is also a spring term, with a stiffness quadratic in the velocity, that represents the minimum compensation term needed to eliminate the possible curvature instability effect (see [3]). In Fig. 1, we represent the operation of parallel transport along the geodesic γ that joins the system position q and the estimate position $\hat{q}$ on the manifold M. We can see for instance that the angle between $S(q,t)$ and $\mathrm{grad}_q F(\hat{q},q)$ is the same as that between their parallel transported counterparts $\mathcal{T}_{//q\to\hat{q}} S(q,t)$ and $\mathcal{T}_{//q\to\hat{q}}\mathrm{grad}_q F(\hat{q},q) = -\mathrm{grad}_{\hat{q}} F(\hat{q},q)$.

B. Invariance on a Tutorial Example

This is just to show that once the gains α and β are chosen, (2) defines a unique observer independent of the choice of a particular set of coordinates on the configuration manifold M.

1) Dynamics in q-Coordinate: We consider the one degree of freedom mechanical system whose Lagrangian is given by

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}q^2$$

which represents the dynamics of the standard oscillator with $q \in \mathbb{R}$: $\ddot{q} = -q$. For this system, as the configuration space is Euclidian, the intrinsic observer equation (2) reads

$$\dot{\hat{q}} = \hat{v} - \alpha(\hat{q} - q)$$
$$\dot{\hat{v}} = -q - \beta(\hat{q} - q). \quad (3)$$

If the gains α and β are chosen positive, we have convergence of $\hat{q}$ and $\hat{v}$ to q and $\dot{q}$.

2) Dynamics in r-Coordinate: Consider now a change of coordinate $r = \exp(q)$. The Lagrangian becomes

$$\mathcal{L}(r,\dot{r}) = \frac{1}{2}\frac{\dot{r}^2}{r^2} - \frac{1}{2}(\ln r)^2$$

and the system dynamics then writes

$$\dot{r} = w$$
$$\dot{w} = \frac{w^2}{r} - r\ln r \qquad \text{for } r \in]0, +\infty[.$$

We are now going to compute the observer (2)

$$\dot{\hat{r}} = \hat{w} - \alpha\,\mathrm{grad}_{\hat{r}} F(r,\hat{r})$$
$$\dot{\hat{w}} = \frac{\hat{w}\dot{\hat{r}}}{\hat{r}} + \mathcal{T}_{//r\to\hat{r}}(-r\ln r) - \beta\,\mathrm{grad}_{\hat{r}} F(r,\hat{r}) + R(\hat{w}, \mathrm{grad}_{\hat{r}} F(\hat{r}, r))\hat{w}. \quad (4)$$

The metric is given by $g = g_{11}$ with $g_{11}(r) = 1/r^2$. The Christoffel symbol is $\Gamma^1_{11}(r) = -(1/r)$. The equation for the geodesic joining r_1 and r_2 is

$$\gamma(s) = \exp\left(\ln\left(\frac{r_2}{r_1}\right)s + \ln r_1\right).$$

We have then $\gamma(0) = r_1$ and $\gamma(1) = r_2$. So, the geodesic distance between r_1 and r_2 is

$$d_G(r_1, r_2) = \int_0^1 \sqrt{g(\gamma(s))(\gamma'(s))^2}\,ds = |\ln r_2 - \ln r_1|$$

where $'$ stands for d/ds. The term $F(\hat{r}, r)$ is then given by

$$F(r,\hat{r}) = \tfrac{1}{2}(\ln r - \ln\hat{r})^2$$

and its gradient by

$$\mathrm{grad}_{\hat{r}} F(r,\hat{r}) = \hat{r}^2\frac{(\ln\hat{r} - \ln r)}{\hat{r}} = \hat{r}(\ln\hat{r} - \ln r).$$

The parallel transport equation along the geodesic

$$s \mapsto \gamma(s) = \exp\left(\ln\left(\frac{\hat{r}}{r}\right)s + \ln r\right)$$

joining r at $s = 0$ and $\hat{r}$ at $s = 1$ reads

$$u' - \frac{1}{\gamma(s)}\gamma'(s)u = 0$$
$$u(0) = -r\ln r$$

for which the solution is given by

$$u(s) = -r\ln r\exp((\ln\hat{r} - \ln r)s).$$

Then, we have

$$\mathcal{T}_{//r\to\hat{r}}(-r\ln r) = u(1) = -\hat{r}\ln r.$$

So, (2) in the r coordinates gives

$$\dot{\hat{r}} = \hat{w} - \alpha\hat{r}(\ln\hat{r} - \ln r)$$
$$\dot{\hat{w}} = \frac{\hat{w}^2}{\hat{r}} - \hat{r}\ln r - \beta\hat{r}(\ln\hat{r} - \ln r).$$

Notice that curvature is zero here. This vanishing is independent of the choice of configuration coordinates, whereas it is false for the Christoffel symbols.

In this set of coordinates, we see that this observer expression is not so intuitive: the error term $\hat{r}(\ln\hat{r} - \ln r)$ is nonlinear and is different from the often used error term $\hat{r} - r$. The convergence is clear since it can be checked that it is just the expression of (3) in r coordinates. When the metric g component is $g_{11}(q) = 1$, we have indeed

$$\mathrm{grad}_{\hat{q}} F(q,\hat{q}) = \hat{q} - q$$
$$\mathcal{T}_{//q\to\hat{q}} S(q,t) = \mathcal{T}_{//q\to\hat{q}}(-q) = -q$$
$$R(\hat{v}, \mathrm{grad}_{\hat{q}} F(\hat{q},q))\hat{v} = 0.$$

So, the observer dynamics (3) and (4) are two expressions of the same observer, written in different configuration coordinate sets.

C. First-Order Approximation

In general, we have no explicit formula for F and $\mathcal{T}_{//q\to\hat{q}}$ once the metric is given. Nevertheless, the curvature terms are explicit

$$\{R(\hat{v}, \mathrm{grad}_{\hat{q}} F(\hat{q},q))\hat{v}\}^i = R^i_{jkl}\hat{v}^k\{\mathrm{grad}_{\hat{q}} F(\hat{q},q)\}^j\hat{v}^l$$

where R^i_{jkl} are the components of the curvature tensor

$$R^i_{jkl} = \frac{\partial\Gamma^i_{jl}}{\partial q^k} - \frac{\partial\Gamma^i_{jk}}{\partial q^l} + \Gamma^i_{pk}\Gamma^p_{jl} - \Gamma^i_{pl}\Gamma^p_{jk}.$$

However, for $\hat{q}$ close to q, F and $\mathcal{T}_{//q\to\hat{q}}$ admits the following approximations:

$$2F = g_{ij}(q)(\hat{q}^i - q^i)(\hat{q}^j - q^j) + O(\|\hat{q} - q\|^3)$$
$$\{\mathrm{grad}_{\hat{q}} F\}^i = \hat{q}^i - q^i + O(\|\hat{q} - q\|^2)$$
$$\{\mathcal{T}_{//q\to\hat{q}} w\}^i = w^i - \Gamma^i_{jl}(q)w^j(\hat{q}^l - q^l) + O(\|\hat{q} - q\|^2)$$

for any w belonging to the tangent space at q to M. The first equality comes from the definition of the geodesic distance. The second one is derived from the definition of the gradient for a scalar function. And the last one is derived from the expression of the parallel transport (see [32], [3] for more precisions). Remark that the "O-terms" will retain their forms when coordinates are changed in a differentiable manner.

Thus, we can construct an explicit approximation of (2) up to order 2. In local coordinates, this gives the following second-order approximate observer that can be integrated numerically:

$$\dot{\hat{q}}^i = \hat{v}^i - \alpha(\hat{q}^i - q^i)$$
$$\dot{\hat{v}}^i = -\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k + S^i(q,t) - \Gamma^i_{jl}(q)S^j(q,t)(\hat{q}^l - q^l) - \beta(\hat{q}^i - q^i) + R^i_{jkl}(q)\hat{v}^k(\hat{q}^j - q^j)\hat{v}^l. \quad (5)$$

In the term $\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k$, it is important to consider $\Gamma^i_{jk}(\hat{q})$ instead of $\Gamma^i_{jk}(q)$ since it is one of the terms of the covariant derivative of $\hat{v}$ with respect to $\dot{\hat{q}}$. Nevertheless in the terms $\Gamma^i_{jl}(q)S^j(q,t)(\hat{q}^l - q^l)$ and $R^i_{jkl}(q)\hat{v}^k(\hat{q}^j - q^j)\hat{v}^l$, we could have used $\Gamma^i_{jk}(\hat{q})$ and $R^i_{jkl}(\hat{q})$, since this represents a second order perturbation. The value of (5) relies on two facts

- the gains are explicit and can be computed via the inertia matrix (g_{ij}) and its q derivatives up to order 2;
- we will prove in the sequel the local convergence of (5) as soon as α and β are strictly positive.

IV. OBSERVER CONVERGENCE

The observer dynamics (2) is locally ($\hat{q} \approx q$) contracting in the sense of [24], [18]: some insight on this property is given in Appendix II. As the system dynamics (1) is a solution of (2), this will give the local convergence.

More precisely, we are going to demonstrate the following result.

Theorem 1: Take (1) defining a dynamical system on the tangent bundle TM. Consider a compact subset K of TM and two positive parameters α and β (the observer gains).

Then, there exist $\varepsilon > 0$ (depending only on K, α and β), $\mu > 0$ and a Riemannian metric G on TM (depending only on α and β) such that, for any solution of (1) remaining in K

$$[0, T[\ \ni t \mapsto X(t) = (q(t), \dot{q}(t)) \in K$$

with $T \le +\infty$, the solution of (2), $t \mapsto \hat{X}(t) = (\hat{q}(t), \hat{v}(t))$ with $\hat{X}(0)$ satisfying $d_G(\hat{X}(0), X(0)) \le \varepsilon$, is defined for all $t \in [0, T[$ and, moreover, $\forall t \in [0, T[$

$$d_G\left(\hat{X}(t), X(t)\right) \le d_G\left(\hat{X}(0), X(0)\right)\exp(-\mu t).$$

Here d_G is the geodesic distance associated to the metric G on TM.

The metric G is, in fact, a modified version of the Sasaki metric [30], i.e., the lift of the kinetic energy metric on TM. The observer gains α and β are involved in the definition of G in order to get the convergence estimation and the fact that, locally, the geodesic distance $d_G(\hat{X}(t), X(t))$ is a decreasing function of t.

Proof: The demonstration follows in two steps.

- For each q, we compute intrinsically (as for the second variation of geodesic) the first variation with respect to $\hat{q}$ and $\hat{v}$ of (2). If we denote by $(\hat{q} + \delta\hat{q}, \hat{v} + \delta\hat{v})$ a point defined in a neighborhood of $(\hat{q}, \hat{v})$, we are looking for the intrinsic formulation of the $(\delta\hat{q}, \delta\hat{v})$-dynamics.

[7] N. Aghannan and P. Rouchon, "An intrinsic observer for a class of lagrangian systems," IEEE Transactions on Automatic Control, vol. 48, no. 6, pp. 936–945, 2003.

Contraction analysis in local coordinates

N. Aghannan and P. Rouchon [7]

- We will deduce from this intrinsic formulation, a metric G on the tangent bundle TM for which the observer dynamics (2) is a contraction for $\hat{q}$ close to q.

A. First Step: First Variation of the Observer Dynamics

We will just mimic here the method that has been used to derive the Jacobi equation [15], that is to say the first variation of the geodesic equation $\nabla_{\dot{q}}\dot{q} = 0$. All the calculations presented here are done in a particular set of coordinates (q^i), but the final results are given in an intrinsic formulation. As the first variation $\delta\hat{v}$ is not an intrinsic term, we are going first to define its intrinsic equivalent ζ that belongs to the tangent space at $\hat{q}$ to M. Then, we are going to determine the dynamics of the intrinsic vectors $(\xi = \delta\hat{q}, \zeta)$ associated to $(\delta\hat{q}, \delta\hat{v})$.

1) Intrinsic Vectors ξ and ζ: We introduce another set of coordinates (an "intrinsic" one)

$$\xi^i = \delta\hat{q}^i$$
$$\zeta^i = \delta\hat{v}^i + \Gamma^i_{jk}(\hat{q})\delta\hat{q}^k\hat{v}^j, \qquad i = 1, \ldots, n. \qquad (6)$$

One can check that the ξ^i and also the ζ^i correspond to the coordinates of two vectors ξ and ζ belonging to the same linear space, the tangent space at $\hat{q}$ to M. Indeed, we have up to second-order terms

- $\xi = \epsilon\dot{\gamma}(0)$, for some small real $\epsilon > 0$ with γ a geodesic that joins the points $\gamma(0) = \hat{q}$ and $\gamma(\epsilon) = \hat{q} + \delta\hat{q}$;
- $\zeta = \mathcal{T}_{//\hat{q}+\delta\hat{q}\to\hat{q}}(\hat{v} + \delta\hat{v}) - \hat{v}$.

The tangent vector ξ and ζ are defined along the curve $t \mapsto \hat{q}(t)$. Thus, we can consider their covariant derivatives $\nabla_{\dot{\hat{q}}}\xi$ and $\nabla_{\dot{\hat{q}}}\zeta$ still belonging to the tangent space at $\hat{q}$ to M.

Notice that the first variation of (2) gives

$$\frac{d}{dt}(\delta\hat{q}^i) = \delta\hat{v}^i - \alpha\{\delta(\mathrm{grad}_{\hat{q}}F(\hat{q}, q))\}^i$$
$$\frac{d}{dt}(\delta\hat{v}^i) = -\delta\left(\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k\right) + \{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t)) - \beta\delta(\mathrm{grad}_{\hat{q}}F(\hat{q},q)) + \delta(R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})\}^i. \qquad (7)$$

2) Computation of $\nabla_{\dot{\hat{q}}}\xi$: In local coordinates we have, for $i = 1, \ldots, n$

$$\begin{aligned}\{\nabla_{\dot{\hat{q}}}\xi\}^i &= \dot{\xi}^i + \Gamma^i_{jk}(\hat{q})\dot{\hat{q}}^k\xi^j \\ &= \delta\hat{v}^i - \alpha\{\delta(\mathrm{grad}_{\hat{q}}F(\hat{q},q))\}^i + \Gamma^i_{jk}(\hat{q})(\hat{v}^k - \alpha\{\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^k)\xi^j \\ &= \delta\hat{v}^i + \Gamma^i_{jk}(\hat{q})\hat{v}^k\xi^j - \alpha\left(\{\delta(\mathrm{grad}_{\hat{q}}F(\hat{q},q))\}^i + \Gamma^i_{jk}(\hat{q})\{\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^k\xi^j\right).\end{aligned}$$

Then, we get

$$\boxed{\nabla_{\dot{\hat{q}}}\xi = \zeta - \alpha\nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q)}$$

with

$$\{\nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^i = \{\delta(\mathrm{grad}_{\hat{q}}F(\hat{q},q))\}^i + \Gamma^i_{jk}(\hat{q})\{\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^k\xi^j.$$

3) Computation of $\nabla_{\dot{\hat{q}}}\zeta$: The covariant derivative of ζ is given by

$$\{\nabla_{\dot{\hat{q}}}\zeta\}^i = \dot{\zeta}^i + \Gamma^i_{jk}(\hat{q})\dot{\hat{q}}^k\zeta^j. \qquad (8)$$

According to (7) and (6), we have

$$\begin{aligned}\dot{\zeta}^i &= \frac{d}{dt}(\delta\hat{v}^i) + \frac{d}{dt}(\Gamma^i_{jk}(\hat{q})\hat{v}^j\xi^k) \\ &= -\delta\left(\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k\right) + \{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t) - \beta\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})\}^i + \frac{d}{dt}(\Gamma^i_{jk}(\hat{q})\hat{v}^j\xi^k).\end{aligned}$$

So, when we consider only the terms of first order in $\delta\hat{q}$ and $\delta\hat{v}$

$$\begin{aligned}\delta\left(\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k\right) &= \frac{\partial\Gamma^i_{jk}(\hat{q})}{\partial\hat{q}^c}\xi^c\hat{v}^j\dot{\hat{q}}^k + \Gamma^i_{jk}(\hat{q})\delta\hat{v}^j\dot{\hat{q}}^k + \Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\xi}^k \\ &= \frac{\partial\Gamma^i_{jk}(\hat{q})}{\partial\hat{q}^c}\xi^c\hat{v}^j\dot{\hat{q}}^k + \Gamma^i_{jk}(\hat{q})\zeta^j\dot{\hat{q}}^k - \Gamma^i_{jk}(\hat{q})\Gamma^j_{pc}(\hat{q})\hat{v}^p\xi^c\dot{\hat{q}}^k + \Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\xi}^k\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\Gamma^i_{jk}(\hat{q})\hat{v}^j\xi^k) &= \frac{\partial\Gamma^i_{jk}(\hat{q})}{\partial\hat{q}^c}\dot{\hat{q}}^c\xi^k\hat{v}^j + \Gamma^i_{jk}(\hat{q})\xi^k\dot{\hat{v}}^j + \Gamma^i_{jk}(\hat{q})\dot{\xi}^k\hat{v}^j \\ &= \frac{\partial\Gamma^i_{jk}(\hat{q})}{\partial\hat{q}^c}\dot{\hat{q}}^c\xi^k\hat{v}^j - \Gamma^i_{jk}(\hat{q})\Gamma^j_{pc}(\hat{q})\xi^k\dot{\hat{q}}^c\hat{v}^p \\ &\quad + \Gamma^i_{jk}(\hat{q})\xi^k\{\mathcal{T}_{//q\to\hat{q}}S(q,t) - \beta\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\}^j + \Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\xi}^k.\end{aligned}$$

Then, we put together the two previous equations

$$\begin{aligned}&\frac{d}{dt}(\Gamma^i_{jk}(\hat{q})\hat{v}^j\xi^k) - \delta\left(\Gamma^i_{jk}(\hat{q})\hat{v}^j\dot{\hat{q}}^k\right) \\ &= -\Gamma^i_{jk}(\hat{q})\zeta^j\dot{\hat{q}}^k + \Gamma^i_{jk}(\hat{q})\xi^k\{\mathcal{T}_{//q\to\hat{q}}S(q,t) - \beta\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\}^j \\ &\quad + \xi^k\dot{\hat{q}}^c\hat{v}^p\left(\frac{\partial\Gamma^i_{pk}(\hat{q})}{\partial\hat{q}^c} - \frac{\partial\Gamma^i_{pc}(\hat{q})}{\partial\hat{q}^k} + \Gamma^i_{jc}(\hat{q})\Gamma^j_{pk}(\hat{q}) - \Gamma^i_{jk}(\hat{q})\Gamma^j_{pc}(\hat{q})\right).\end{aligned}$$

So, we get for $\dot{\zeta}^i$

$$\begin{aligned}\dot{\zeta}^i &= -\Gamma^i_{jk}(\hat{q})\zeta^j\dot{\hat{q}}^k + \{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t) - \beta\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})\}^i \\ &\quad + \Gamma^i_{jk}(\hat{q})\xi^k\{\mathcal{T}_{//q\to\hat{q}}S(q,t) - \beta\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\}^j - \xi^k\dot{\hat{q}}^c\hat{v}^pR^i_{kcp}(\hat{q})\end{aligned}$$

which gives the following "semi-intrinsic" expression

$$\nabla_{\dot{\hat{q}}}\zeta = -R\left(\dot{\hat{q}}, \xi\right)\hat{v} + \nabla_\xi(\mathcal{T}_{//q\to\hat{q}}S(q,t)) - \beta\nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + \text{“}\nabla_\xi\text{”}(R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})$$

where

$$\{\nabla_\xi(\mathcal{T}_{//q\to\hat{q}}S(q,t))\}^i = \{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t))\}^i + \Gamma^i_{jk}(\hat{q})\xi^k\{\mathcal{T}_{//q\to\hat{q}}S(q,t)\}^j$$

and

$$\{\text{“}\nabla_\xi\text{”}(R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})\}^i = \{\delta(R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v})\}^i + \Gamma^i_{jk}(\hat{q})\xi^k\{R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\}^j.$$

For the curvature term, we use "∇_ξ" instead of ∇_ξ since this is not a true covariant derivation with respect to ξ: the vector $R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}$ is only defined along the curve $t \mapsto \hat{q}(t)$. Thus, we cannot define properly its covariant derivative in a direction ξ that is not colinear to $\dot{\hat{q}}$.

Let us express now the "∇_ξ" term in an intrinsic way. Since

$$\delta\hat{v}^i = \zeta^i - \Gamma^i_{jk}(\hat{q})\delta\hat{q}^k\hat{v}^j$$
$$\{\delta(\mathrm{grad}_{\hat{q}}F(\hat{q},q))\}^i = \{\nabla_\xi\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^i - \Gamma^i_{jk}(\hat{q})\delta\hat{q}^k\{\mathrm{grad}_{\hat{q}}F(\hat{q},q)\}^j$$

standard computations provide the following intrinsic expression:

$$\begin{aligned}&\text{“}\nabla_\xi\text{”}(R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}) \\ &= (\nabla_\xi R)(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v} + R(\zeta, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v} + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\zeta + R(\hat{v}, \nabla_\xi\mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\end{aligned}$$

where $\nabla_\xi R$ is the covariant derivative of the curvature tensor along ξ.

Finally, we have the following intrinsic formula for the first variation of (2) with respect to $\hat{q}$ and $\hat{v}$:

$$\boxed{\begin{aligned}\nabla_{\dot{\hat{q}}}\xi &= \zeta - \alpha\nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) \\ \nabla_{\dot{\hat{q}}}\zeta &= -R\left(\dot{\hat{q}}, \xi\right)\hat{v} + \nabla_\xi(\mathcal{T}_{//q\to\hat{q}}S(q,t)) \\ &\quad -\beta\nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) + (\nabla_\xi R)(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v} \\ &\quad + R(\zeta, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v} + R(\hat{v}, \mathrm{grad}_{\hat{q}}F(\hat{q},q))\zeta \\ &\quad + R(\hat{v}, \nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q))\hat{v}\end{aligned}} \qquad (9)$$

where ξ corresponds to the variation of $\hat{q}$. When $S = 0, F = 0$ and when we set to zero the curvature terms in (2), we recover the classical Jacobi equation:

$$\frac{D^2\xi}{Dt^2} = -R\left(\dot{\hat{q}}, \xi\right)\dot{\hat{q}}$$

where $D/Dt = \nabla_{\dot{\hat{q}}}$.

4) First Variation When $\hat{q}$ Is Close to q: Assume that $\hat{q}$ is close to q (we do not assume here that $\hat{v}$ is close to v). Then, the aforementioned first variation becomes much simpler since up to order 1 in $\hat{q} - q$

$$\mathrm{grad}_{\hat{q}}F(\hat{q},q) = 0, \quad \nabla_\xi\,\mathrm{grad}_{\hat{q}}F(\hat{q},q) = \xi \quad \dot{\hat{q}} = \hat{v}.$$

Furthermore

$$\{\nabla_\xi(\mathcal{T}_{//q\to\hat{q}}S(q,t))\}^i = \{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t))\}^i + \Gamma^i_{jk}(\hat{q})\{\mathcal{T}_{//q\to\hat{q}}S(q,t)\}^k\xi^j.$$

As

$$\{\mathcal{T}_{//q\to\hat{q}}S(q,t)\}^i = S(q,t)^i - \Gamma^i_{jk}(q)(\hat{q}^j - q^j)S(q,t)^k$$

for q close to $\hat{q}$, we then have

$$\{\delta(\mathcal{T}_{//q\to\hat{q}}S(q,t))\}^i = -\Gamma^i_{jk}(q)\xi^jS(q,t)^k$$
$$\Gamma^i_{jk}(\hat{q})\{\mathcal{T}_{//q\to\hat{q}}S(q,t)\}^k\xi^j = \Gamma^i_{jk}(q)\xi^jS(q,t)^k$$

up to terms of order 1 in $(\hat{q} - q)$. So

$$\nabla_\xi(\mathcal{T}_{//q\to\hat{q}}S(q,t)) = 0.$$

Thus, for $q = \hat{q}$, (9) becomes

$$\boxed{\begin{aligned}\nabla_{\dot{\hat{q}}}\xi &= \zeta - \alpha\xi \\ \nabla_{\dot{\hat{q}}}\zeta &= -\beta\xi.\end{aligned}} \qquad (10)$$

In a certain sense, we recover the Euclidian case with the classical Luenberger observer described in the introduction. This is due to the cancellation of the curvature terms.

B. Second Step: Contraction Analysis

Let us prove first that (10) implies that the dynamics is strictly contracting when $\hat{q} = q$. Elementary continuity arguments show that contraction remains for $\hat{q}$ close to q. This explains the constraint on the initial condition for the observer dynamics. To speak of contraction, we first need to define a metric on the observer state space, i.e., on the tangent bundle TM.

1) Riemannian Structure on the Tangent Bundle TM: Since the observer gains α and β are positive, the matrix

$$A = \begin{pmatrix} -\alpha & 1 \\ -\beta & 0 \end{pmatrix}$$

is Hurwitz and there exists a symmetric positive–definite matrix Q such that

$$A'Q + QA = -I.$$

Set

$$Q = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

and consider the quantity

$$V(\xi, \zeta) = \frac{a}{2}\langle\xi, \xi\rangle + c\langle\xi, \zeta\rangle + \frac{b}{2}\langle\zeta, \zeta\rangle \qquad (11)$$

where $\langle,\rangle$ is the scalar product associated to the metric g deduced from the kinetic energy. This quantity endows TM with a metric, since Q is positive definite: in local coordinates (q^i, v^i), the length of the small vector $(\delta q^i, \delta v^i)$ tangent to (q, v) at TM is

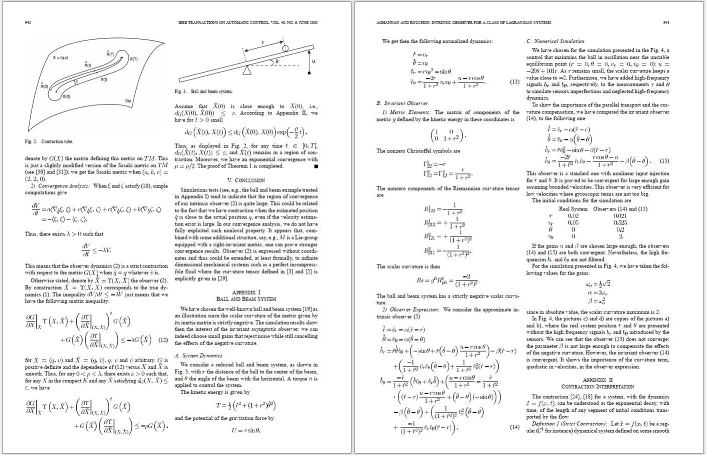
$$\begin{aligned}&V\left(\delta q, (\delta v^i + \Gamma^i_{kl}(q)v^k\delta q^l)_{i=1\cdots n}\right) \\ &= \frac{a}{2}g_{ij}\delta q^i\delta q^j + cg_{ij}(\delta v^i + \Gamma^i_{kl}(q)v^k\delta q^l)\delta q^j \\ &\quad + \frac{b}{2}g_{ij}(\delta v^i + \Gamma^i_{kl}(q)v^k\delta q^l)(\delta v^j + \Gamma^j_{kl}(q)v^k\delta q^l).\end{aligned}$$

In the local coordinates (q^i, v^i), the metric is a $2n \times 2n$ matrix with entries function of q and v. Using notation $X = (q, v)$, we

[7] N. Aghannan and P. Rouchon, "An intrinsic observer for a class of lagrangian systems," IEEE Transactions on Automatic Control, vol. 48, no. 6, pp. 936–945, 2003.

Contraction analysis in local coordinates

N. Aghannan and P. Rouchon [7]



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Intrinsic LES analysis = (Contraction analysis)

Observer: $\begin{cases} \dot{\hat{q}} = \hat{v} - \alpha \nabla F(\hat{q}, q) \\ \nabla_{\dot{\hat{q}}} \hat{v} = -\beta \nabla F(\hat{q}, q) + R(\hat{v}, \nabla F)\hat{v} \end{cases}$

Rewrite the observer as

$$\nabla_{\dot{\hat{q}}} \dot{\hat{q}} = -\alpha \nabla_{\dot{\hat{q}}} \nabla F - \beta \nabla F + R(\dot{\hat{q}}, \nabla F)(\dot{\hat{q}} + \alpha \nabla F) \tag{36}$$

$\Longrightarrow q(\cdot)$ a solution to (36).

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Task 2

Analyze the LES of $q(\cdot)$

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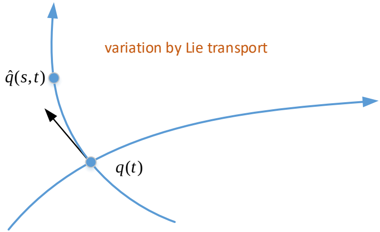
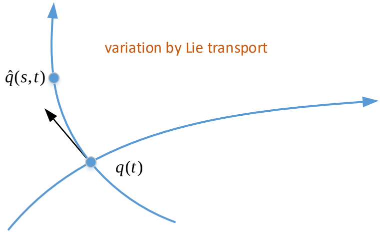
LES analysis of $q(\cdot)$



$$\nabla_{\dot{\hat{q}}}\dot{\hat{q}} = -\alpha\nabla_{\dot{\hat{q}}}\nabla F - \beta\nabla F + R(\dot{\hat{q}}, \nabla F)(\dot{\hat{q}} + \alpha\nabla F) \tag{37}$$



$$\nabla_{\dot{\hat{q}}}\dot{\hat{q}} = -\alpha\nabla_{\dot{\hat{q}}}\nabla F - \beta\nabla F + R(\dot{\hat{q}}, \nabla F)(\dot{\hat{q}} + \alpha\nabla F) \quad (37)$$

The covariant derivative in the direction of $\hat{q}'$ (the Lie transport):

$$\begin{aligned}
\nabla_{\hat{q}'}\nabla_{\dot{\hat{q}}}\dot{\hat{q}} =& -\alpha\nabla_{\hat{q}'}\nabla_{\dot{\hat{q}}}\nabla F - \beta\nabla_{\hat{q}'}\nabla F + \nabla_{\hat{q}'}[R(\dot{\hat{q}}, \nabla F)(\dot{\hat{q}} + \alpha\nabla F)] \\
=& -\alpha\nabla_{\dot{\hat{q}}}\nabla_{\hat{q}'}\nabla F - \alpha R(\dot{\hat{q}}, \hat{q}')\nabla F - \beta\nabla_{\hat{q}'}\nabla F \\
& + \nabla_{\hat{q}'}[R(\dot{\hat{q}}, \nabla F)(\dot{\hat{q}} + \alpha\nabla F)] \\
=& -\alpha\nabla_{\dot{\hat{q}}}\hat{q}' - \beta\hat{q}' + R(\dot{\hat{q}}, \nabla_{\hat{q}'}\nabla F)\dot{\hat{q}} \\
=& -\alpha\nabla_{\dot{\hat{q}}}\hat{q}' - \beta\hat{q}' + R(\dot{\hat{q}}, \hat{q}')\dot{\hat{q}},
\end{aligned}$$

LES analysis of $q(\cdot)$

The above calculation results in

$$\frac{D^2\hat{q}'}{dt^2} + \alpha\frac{D\hat{q}'}{dt} + \beta\hat{q}' = 0 \tag{38}$$

$\hat{q}'$ the Lie transport along $q(t)$, $\frac{D}{dt}$ the covariant derivative along $q(t)$. Equation (38) has the following structure

$$\ddot{x} + \alpha\dot{x} + \beta x = 0$$

where $\alpha, \beta > 0$, which is ES.

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Equip TM with a variant of Sasaki metric

$$\Downarrow$$

Complete lift system along $q(\cdot)$ is ES

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Complete lift system along $q(\cdot)$ is ES

$$d((q,v),(\hat{q},\hat{v}))(t) \le k d((q,v),(\hat{q},\hat{v}))(0)e^{-\lambda t}$$

Conclusion

Conclusion

- A geometric framework for contraction analysis: fundamental theorems, novel characterizations, connection to Lyapunov theory.

Conclusion and perspective

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- Study contraction related practical examples: extremum seeking, synchronization, observers, robustness of NHIM etc.

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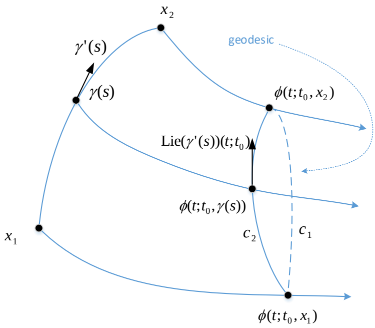
- Differential positive system, monotone systems.

FIN

Questions?

Backup slides

Sketch of the proof



- c_1: geodesic from $\phi(t;t_0,x_1)$ to $\phi(t;t_0,x_2)$
- c_2: flow of the geodesic γ

$$t \mapsto \big(\phi(t;t_0,\gamma(s)), \text{Lie}(\gamma'(s))(t;t_0)\big) \in TM$$

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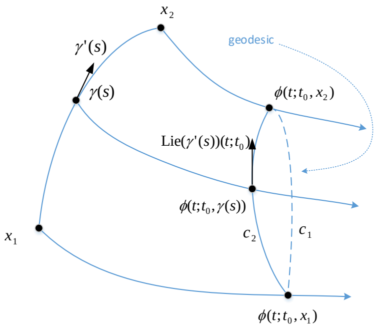

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solution to the CL lift system

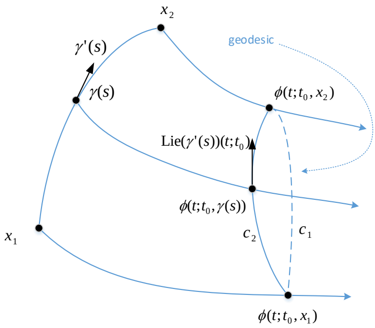
$+$

$\mathcal{L}_{\tilde{f}} V \leq -\alpha(V)$

$\Downarrow$

$$V(t, \mathrm{Lie}(\gamma'(s))(t;t_0)) \leq \beta(V(t_0,\gamma'(s)), t-t_0) \tag{39}$$

Sketch of the proof



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$$d(\phi(t;t_0,x_1),\phi(t;t_0,x_2)) \leq \ell(c_2)$$

$$\leq \int_0^\ell |\text{Lie}(\gamma'(s))(t;t_0)| ds$$

$$\leq \int_0^\ell \alpha_1^{-1}(V(t,\text{Lie}(\gamma'(s))(t;t_0)))ds \tag{40}$$

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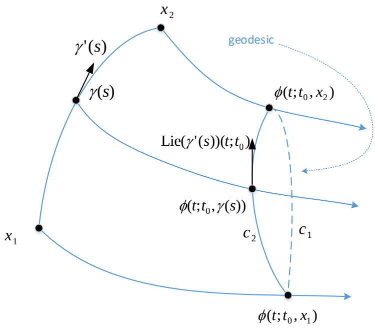
Sketch of the proof



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(39) + (40) $\Rightarrow$ IAS

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Finsler-Lyapunov function: a relaxation

Definition (D. Wu *et al.* 2021)

Given a Finsler structure F on M, a candidate Finsler-Lyapunov function on $U \subseteq M$ is a C^1 function $V : \mathbb{R}_+ \times TM \to \mathbb{R}_+$ satisfying

$$\alpha_1(F(x,\delta x)) \le V(t,x,\delta x) \le \alpha_2(F(x,\delta x)), \quad \forall (t,x,\delta x) \in \mathbb{R}_+ \times TM|_U \tag{41}$$

where α_1, α_2 are $\mathcal{K}_\infty$ functions.

Remarks

- On Riemannian manifolds, $F(x,\delta x) = |\delta x|_x$
- In F. Forni and R. Sepulchre 2014, α_1, α_2 are $\alpha_i(s) = c_i s^p$ for $p \ge 1$.