# Minimum Energy to send $k$ bits with and without feedback

**A nice result of Polyanskiy, Poor and Verdu (NEWS FLASH) + an attempt at improvement**
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# Disclaimer

This is mainly presentation of work in

“Minimum energy to send k bits with and without feedback”, Polyanskiy, Poor, Verdu, ISIT 2010, Austin, Texas, U.S.A., June 13 - 18, 2010

I have been fascinated by this paper for over a year, and talked with some of you about it

I will also show some results I obtained this morning while preparing this presentation.

And then I will show what I later found

# Classical Asymptotic Results

Want to transmit information bits over a discrete time AWGN channel

$$y = x + z, \qquad z_k \sim N(0, N_0/2)$$

Message of $k$ bits coded into the possibly infinite sequence $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$

Output sequence $y = (y_1, y_2, \ldots)$

Allowed block error probability $\epsilon$

Energy measure $\sum x_k^2$

# Shannon's Classical Result

**Achievable Rate** In the limit $\epsilon \to 0$, $k \to \infty$, the smallest achievable energy per bit $E_b = \frac{E}{k}$ is

$$\min \frac{E_b}{N_0} = \log_e 2 = -1.59 dB$$

where $N_0/2$ is the noise power per degree of freedom

Random block coding (with long blocks). Central limit theorem.

**Fact: A noise-free (causal) feedback channel does not help, same performance!**

What if a finite number of bits need to be transmitted?

(And what if we have a max delay constraint?)

# Block code without feedback

An $(E, M, \epsilon)$ code is a list of code words

$$(c_1, \ldots, c_M) \in (\mathbb{R}^\infty)^M$$

satisfying

$$\|c_j\|^2 \leq E, \quad j = 1, \ldots, M$$

and a decoder $g$ satisfying

$$P[g(y) \neq W] \leq \epsilon$$

where $y$ is the response to $x = c_W$

Energy per bit if we send $k$ bits

$$E_b(k, \epsilon) = \frac{1}{k} \inf\{E : \exists (E, 2^k, \epsilon) - \text{code}\}$$

# Block Code with feedback

Similar to above, but where the encoder function has the form

$$X_k = f_k(W, Y_1^{k-1})$$

The encoder hence has (perfect) information about the past channel outputs

Energy restriction

$$E(\|x\|^2 \mid W = j) \leq E, \quad \forall j$$

$$E_f(k, \epsilon) = \frac{1}{k} \inf\{E : \exists (E, 2^k, \epsilon) - \text{feedback code}\}$$

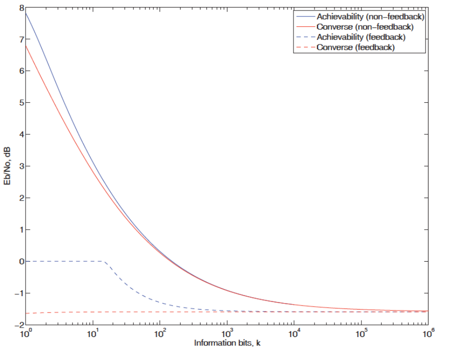
# Results

Yury Polyanskiy, Vincent Poor, Sergio Verdu recently showed upper and lower bounds for the needed energy in the finite regime

Upper bounds: A coder and decoder is found giving a certain performance

Lower bounds: Information theory is used to prove that you can't do better

# Finite Block Length Performance




**Huge improvement by feedback for finite block lengths !**

Open question: Is the lower red line achievable with feedback?

**Can one transmit one bit of information over the AWGN channel using energy = ln(2) = -1.59dB ??**

# Notation

Area to the right of $x$ of a Gaussian normal distribution

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \mathrm{Prob}(Z_n > x)$$

# No Feedback - Upper and Lower Bound

There exists an $(E, 2^k, \epsilon)$ code with

$$\epsilon = 1 - \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \left[ 1 - Q\left( \frac{z}{\sqrt{N_0/2}} \right) \right]^{2^k - 1} e^{-\frac{(z-\sqrt{E})^2}{N_0}} dz$$

Proof: Use $x = \sqrt{E} e_i$ and ML-decoder $i = \operatorname{argmax}_i \, y_i$

Any $(E, 2^k, \epsilon)$ code without feedback satisfies

$$\frac{1}{2^k} \geq Q\left( \sqrt{\frac{2E}{N_0}} + Q^{-1}(1 - \epsilon) \right)$$

Proof: More difficult, but not extremly hard

# Feedback - Lower Bound

Shannon's asymptotic result for $k \to \infty$, $\epsilon \to 0$

$$\frac{E_b}{N_0} \geq \ln(2) = -1.59 \text{ dB}$$

is obviusly a valid bound for all finite $k$ as well.

If error probability $\epsilon > 0$ is allowed one can instead prove the slightly weaker lower bound

$$\frac{E_b}{N_0} \geq \ln(2) d(1 - \epsilon \| 2^{-k})$$

where $d(x\|y) = x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$ is the binary relative entropy

# Feedback - Upper Bound

How should one construct a feedback scheme that improves decoding?

Idea: Use more transmit energy if the decoder is on the wrong track

Consider the case with $k = 1$ bit of information to be transmitted

The following scheme achieves error-free transmission with

$$\frac{E}{N_0} = 1 = 0 \text{ dB}$$

using an ML-decoder:

# Transmitting one bit of information with $E = N_0$

Assume $W = \pm 1$ and choose an arbitrary constant $d > 0$

At time $n$ transmit

$$x_n = \begin{cases} Wd & P(W \mid Y_1^{n-1}) \leq P(-W \mid Y_1^{n-1}) \\ 0 & \text{otherwise} \end{cases}$$

If W=1 we will hence use $x_n = u$ if the ML-decoder is on the wrong track, else $x_n = 0$

Why does the scheme give $\epsilon = 0$ and expected energy $E = N_0$?

# Sketch of Proof

Log-likelihood ratio

$$S_n = \log \frac{P(W = +1 \mid Y^n)}{P(W = -1 \mid Y^n)}$$

For $W = \pm 1$

$$S_n = S_{n-1} + dZ_n \pm \frac{1}{2} d^2 \qquad \text{(exercise)}$$

When $u \to 0$, $S_n$ hence behaves like a random-walk with a positive or negative trend

With $B_t$ Brownian motion we have

$$S_n = \left( \frac{t}{2} + \sqrt{\frac{N_0}{2}} B_t \right)_{t=nd^2}$$

Random walk with positive trend if $W = 1$, negative if $W = -1$

# Analysis

Amount of energy spent = Average time $S_n$ is spent below zero

$$
\begin{aligned}
E(T) &= \int_0^\infty P\left(\frac{t}{2}+\sqrt{\frac{N_0}{2}}B_t \le 0\right) dt = \\
&= \int_{t=0}^\infty Q\left(\frac{t/2}{\sqrt{N_0 t/2}}\right) dt \\
&= \int\int_{0\le t\le 2x^2 N_0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{x=0}^\infty 2x^2 N_0 e^{-x^2/2} dx = N_0
\end{aligned}
$$

A similar scheme give another upper bound (see figure) when $k > 1$

# Open Questions

To beat this performance, one should come up with a more clever feedback scheme.

- Can you find a feedback scheme that achieves the magical -1.59dB energy to transmit a single bit?
- What if the feedback channel has limited capacity?
- What if we have finite time available and decoding must be achieved before a certain deadline?

# News Flash

PPV uses a relay controller of the log-likelihood ratio $S$

$$u_1(S) = d\,1_{S\le 0} \quad \text{and} \quad u_{-1}(S) = -u_1(-S)$$

The only thing used in the proof is really that

$$u_1(S) + u_1(-S) = d \quad \text{and} \quad u_{-1}(S) = -u_1(-S) \qquad (1)$$

Proof: If $W = 1$ we get (for $N_0 = 1$)

$$\begin{aligned} S_n - S_{n-1} &= -\frac{1}{2}\left[(Y_n - u_1(S))^2 - (Y_n - u_{-1}(S))^2\right] \\ &\quad [\text{ Use } Y_n = u_1(S) + Z_n \text{ and } (1)] \\ &= -\frac{1}{2}\left[Z_n^2 - (d + Z_n)^2\right] \\ &= dZ_n + \frac{d^2}{2} \end{aligned}$$

Same as before! We can optimize over all $u_1(S)$ fullfilling (1)

# BoB's Optimization

We want to find $u(S)$ that minimizes the spent energy

If $N_0 = 1$ it is given by

$$
\begin{aligned}
E &= \int_{t=0}^{\infty}\int_{x=-\infty}^{\infty} u^2(x)\frac{1}{\sqrt{\pi t}}e^{-\frac{(x-t/2)^2}{t}}dxdt \\
&= \int_{x=0}^{\infty} u^2(x)f(x) + (1-u(x))^2 f(-x)dx
\end{aligned}
$$

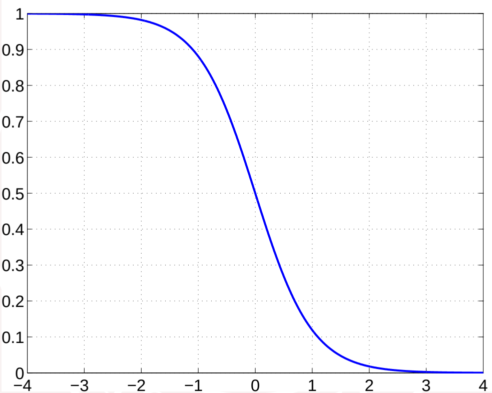
where

$$
f(x) = \int_{t=0}^{\infty}\frac{1}{\sqrt{\pi t}}e^{-\frac{(x-t/2)^2}{t}}dt = 2\min(e^{2x},1)
$$

The optimal controller is given by

$$
u(x) = \frac{f(-x)}{f(x)+f(-x)} = \frac{e^{-x}}{e^{x}+e^{-x}}
$$

# Optimal Controller



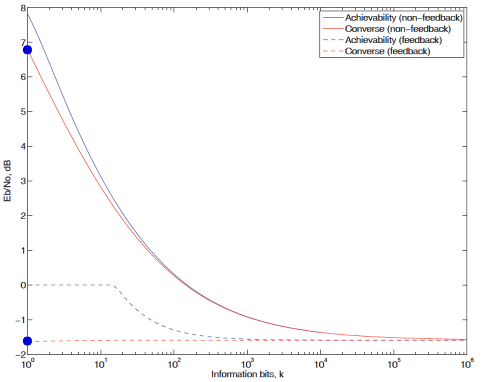
# Resulting Optimized Performance

With this $u(x)$ we get the energy

$$
\begin{aligned}
E &= \int_{x=-\infty}^{\infty} u^2(x) 2\min(e^{2x},1)dx \\
&= \int_{x=-\infty}^{0} \left(\frac{e^{-x}}{e^{x}+e^{-x}}\right)^2 2e^{2x}dx + \int_{x=0}^{\infty} \left(\frac{e^{-x}}{e^{x}+e^{-x}}\right)^2 2dx \\
&= \ln(2)
\end{aligned}
$$

This means we have closed the gap between the upper and lower bound and solved the outstanding open question in PPV!

# Optimal Controller



# News Flash 2

While looking for the address to one of the authors, after finding this nice result, I found a link to a new journal publication I was not aware of:

Y. Polyanskiy, H. V. Poor and S. Verdú, "Minimum energy to send k bits through the Gaussian channel with and without feedback," IEEE Trans. Inf. Theory, vol. 57, no. 8, pp. 4880 - 4902, Aug. 2011, **submitted May 5 2011**.

This article contains the result I just sketched :-(

# Conclusions

- Feedback improves performance for finite block length making the Shannon asymptotic AWGN performance even for a block of one bit
- Many interesting problems with imperfect feedback channel and/or delay constraints are still open
- Always check the recent literature
- Do not postpone giving internal seminars