(Almost) Global UAV Control System Design

Aided by Lyapunov, Barbalat, and Matrosov

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Introduction

Difficulties

- Nonlinear
- Nonautonomous
- Configuration manifolds

Main idea

- Example 1 - Peaking
- Example 2 - Uniform stability
- Example 3 - Case study

Figure 1: The small Crazyflie quadrotor

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Figure 1: The small Crazyflie quadrotor

Pontus: Good idea to talk about Lyapunov functions!

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- Example 1 - Peaking
- Example 2 - Uniform stability
- Example 3 - Case study [1]

Figure 1: The small Crazyflie quadrotor

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[1] E. Lefeber, M. Greiff, and A. Robertsson, "Filtered output feedback tracking control of a quadrotor UAV," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 5764–5770, 2020

Overview and main thread of presentation

- Introduction
  - Pitfalls in NLTV analysis
  - Introducing the case study
- Lyapunov's Second Method
  - The main idea
- Barbălat's Lemma
  - The main idea
  - Useful variations
  - Application
- Matrosov's Theorems
  - The main idea
  - Application
- Simulation example

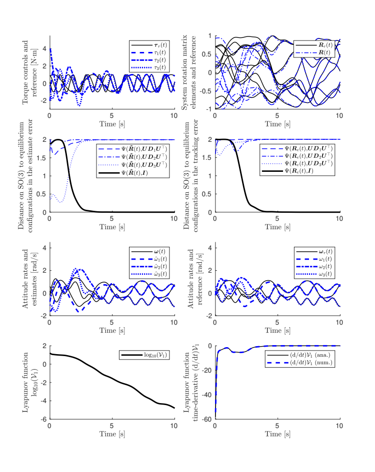
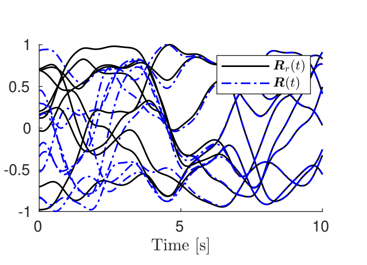


Figure 2: Example simulation (to be explained)

Introduction - A note of caution

Some warnings

- Quite dense
- Lots of signals...!
- Some omitted details



- Σ A system (with memory)
- $\boldsymbol{R}$ A rotation (always $\in \mathbb{R}^{3\times 3}$)
- $\cdot_r$ A reference
- $\cdot_e$ A tracking error
- G Global, as in *globally* stable (GS)
- A Asymptotic, as in *asymptotically* stable (AS)
- E Exponential, as in *exponentially* stable (ES)
- U Uniform, as in *uniformly* stable (US)

Introduction - Curious Example 1

Consider two linear systems

$$\Sigma_1' : \quad \dot{\boldsymbol{x}}_1 = \boldsymbol{A}_1 \boldsymbol{x}_1, \qquad \boldsymbol{x}_1(t_\circ) = \boldsymbol{x}_{1\circ} \tag{1a}$$

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If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is AS, as

$$\{\Sigma_1, \Sigma_2\} : \quad \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \tag{3}$$

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What if the systems are nonlinear and non-autonomous?

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Consider two nonlinear systems

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Consider the cascade

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If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is...

...it depends!

Introduction - Curious Example 1

Example (Peaking)

Let $t_\circ = 0$, and consider a nonlinear system

$$\Sigma_1' : \dot{x}_1 = -x_1^3, \qquad x_1(t_\circ) = x_{1\circ},$$
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The solution for the system $\{\Sigma_1, \Sigma_2\}$ *is*

$$x_1(t) = \mathrm{sign}(x_{1\circ})(x_{1\circ}^{-2} + 2x_{2\circ}(e^{-t} - 1) + 2t)^{-1/2}$$
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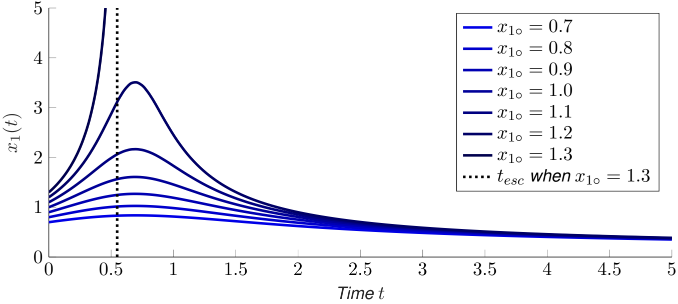
Example (Peaking)

Here Σ_1' is GAS, and Σ_2 is GAS, but for their cascade through g, the solution $x_1(t)$ diverges with a finite escape time even for $x_{1\circ} > 0$.



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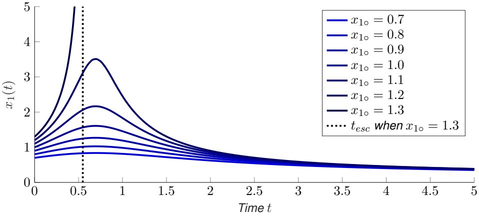

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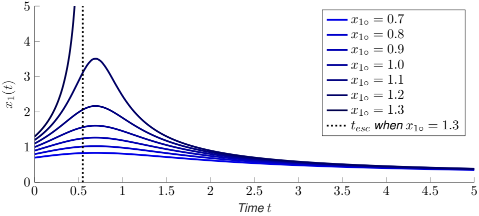

If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is...

GAS if a set of sufficient conditions on $\{\Sigma_1', \Sigma_2\}$ and g are met.

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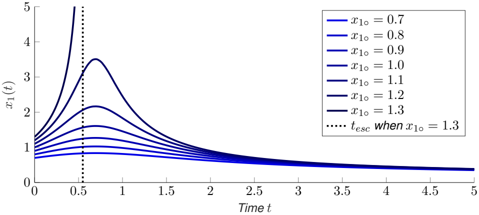

Growth rate: If $\{\Sigma_1', \Sigma_2\}$ is AS, and there exists continuous $\theta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\|g(x_1, x_2)\| \leq \theta_1(\|x_2\|) + \theta_2(\|x_2\|)\|x_1\|$, then $\{\Sigma_1, \Sigma_2\}$ is also AS (see e.g. Panteley '99 [2], or Loria '05 [3]).

Introduction - Curious Example 1

Takeaway

In the nonlinear setting, the separation principle generally does not apply. Care must be taken when connecting a found controller with an observer, as the introduced dynamics may cause the states to diverge, even if $\{\Sigma_1', \Sigma_2\}$ *has very nice properties. Especially true when aiming for global or almost global stability properties.*

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In the nonlinear setting, the separation principle generally does not apply. Care must be taken when connecting a found controller with an observer, as the introduced dynamics may cause the states to diverge, even if $\{\Sigma_1', \Sigma_2\}$ *has very nice properties. Especially true when aiming for global or almost global stability properties.*

With a ”nice” feedback, a ”nice” estimator and ”good” connection, are asymptotic stability properties enough? What about robustness?

Introduction - Curious Example 2

Consider two *non-autonomous* systems:

$$\Sigma: \quad \dot{\boldsymbol{x}} = f(t, \boldsymbol{x}),$$
$$\Sigma_\Delta: \quad \dot{\boldsymbol{x}} = f(t, \boldsymbol{x}) + \boldsymbol{\Delta}(t, \boldsymbol{x}),$$

where $\|\boldsymbol{\Delta}(t, \boldsymbol{x})\| \leq L$ for all $t \geq t_\circ$. What does Σ say about on Σ_Δ?

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Example (Loria, Panteley, Teel ’99)

Consider a nominal system with defined by $a(t) = (t+1)^{-1}$*, with*

$$\dot{x} = f(t, x) = \begin{cases} -a(t)\mathrm{sign}(x) & \text{if } \ |x| > a(t) \\ -x & \text{if } \ |x| \leq a(t) \end{cases}, \quad t \geq t_\circ \geq 0.$$

When adding a $\Delta(t) = L \neq 0$*, solutions grow unbounded as* $t \to \infty$*. In fact,* $\lim_{t\to\infty} x(t)/t = \pm L$ *(depending on the sign of* $x(t_\circ)$*)*

Introduction - Curious Example 2

Problem: the solution of $x(t)$ for the unperturbed system $\dot{x} = f(t, x)$ depends on $(t_\circ, x(t_\circ))$, the convergence to the origin is not *uniform*.

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Solution: Require *uniform* asymptotic stability (independent of $t_\circ$).

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Problem: the solution of $x(t)$ for the unperturbed system $\dot{x} = f(t, x)$ depends on $(t_\circ, x(t_\circ))$, the convergence to the origin is not *uniform*.

Solution: Require *uniform* asymptotic stability (independent of $t_\circ$).

Takeaway

In general, to say something about the robustness properties of systems on the form $\Sigma : \dot{\boldsymbol{x}} = f(t, \boldsymbol{x})$ we require uniform stability properties. Several local or global boundedness results follow (see e.g., Khalil '96 [4, Theorem 3.18 combined with Lemma 4.3]).

Introduction

The attitude dynamics of the UAV

$$\Sigma : \begin{cases} \dot{\boldsymbol{R}} &= \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega}), \\ \boldsymbol{J}\dot{\boldsymbol{\omega}} &= \boldsymbol{S}(\boldsymbol{J}\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}, \end{cases} \qquad \text{(Controlled system)} \tag{6a}$$

$$\Sigma_r : \begin{cases} \dot{\boldsymbol{R}}_r &= \boldsymbol{R}_r\boldsymbol{S}(\boldsymbol{\omega}_r), \\ \boldsymbol{J}\dot{\boldsymbol{\omega}}_r &= \boldsymbol{S}(\boldsymbol{J}\boldsymbol{\omega}_r)\boldsymbol{\omega}_r + \boldsymbol{\tau}_r, \end{cases} \qquad \text{(Reference system)} \tag{6b}$$

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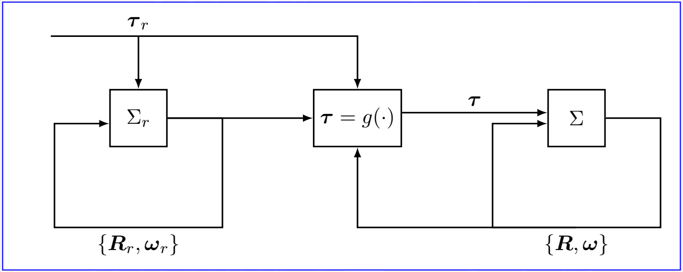
Introduction

Objective: Find $g(\boldsymbol{R}, \boldsymbol{\omega}, \boldsymbol{R}_r, \boldsymbol{\omega}_r, \boldsymbol{\tau}_r)$ such that $\boldsymbol{R} \to \boldsymbol{R}_r, \boldsymbol{\omega} \to \boldsymbol{\omega}_r$



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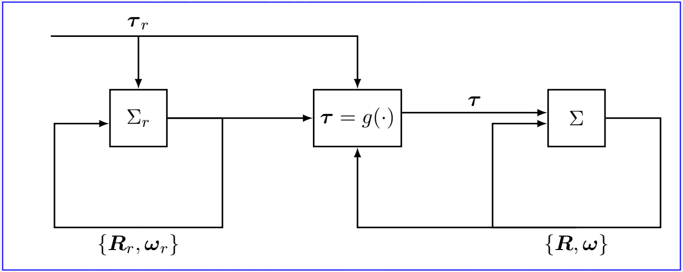

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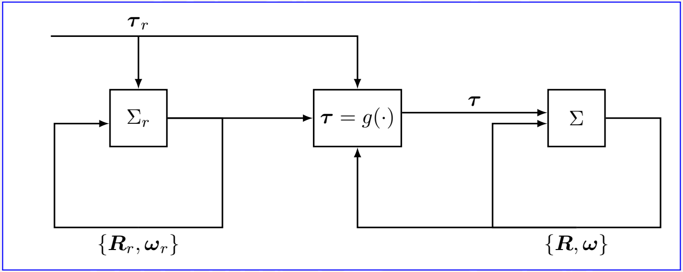

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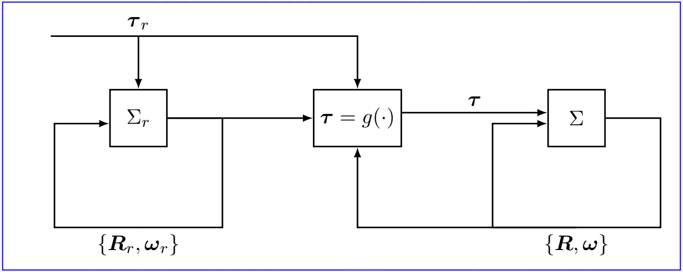
- Essentially a full-state feedback - requires an estimator

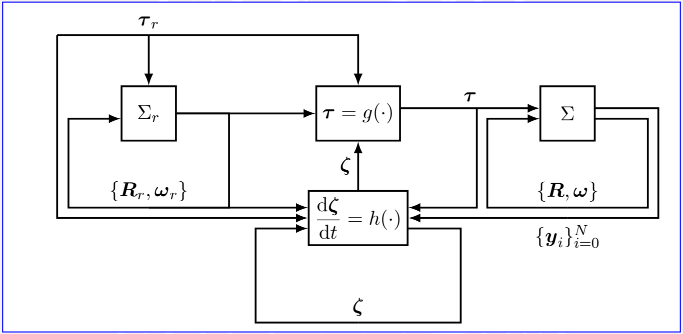
Introduction

Objective: Find $g(\boldsymbol{R}, \boldsymbol{\omega}, \boldsymbol{R}_r, \boldsymbol{\omega}_r, \boldsymbol{\tau}_r)$ such that $\boldsymbol{R} \to \boldsymbol{R}_r, \boldsymbol{\omega} \to \boldsymbol{\omega}_r$



- Many (almost) global solutions exist [5]–[8] , however...
- Essentially a full-state feedback - requires an estimator
- Stability should to be uniform, estimator needs to be (almost) globally stabilizing, interconnection needs to satisfy conditions.

Alternatively, solve a *filtered output feedback problem*, as in [1].



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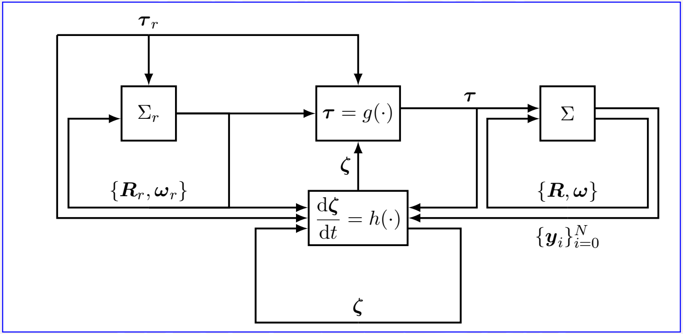

- Define a filter memory $\boldsymbol{\zeta}$ (here $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\omega}}\} \in \mathrm{SO}(3) \times \mathbb{R}^3$).

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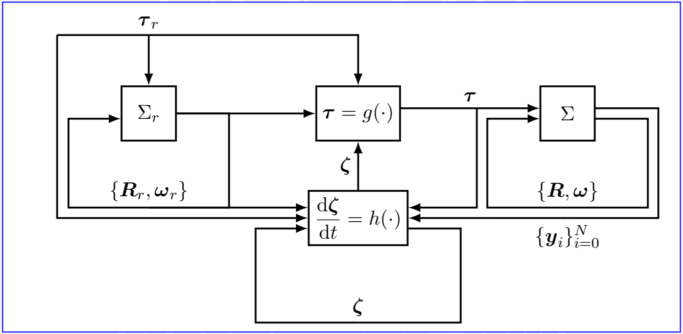

- Define a filter memory ζ (here $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\omega}}\} \in \mathrm{SO}(3) \times \mathbb{R}^3$).
- Define an update of ζ in a set of measurements $\{\boldsymbol{y}_i\}_{i=1}^N$ (IMU).

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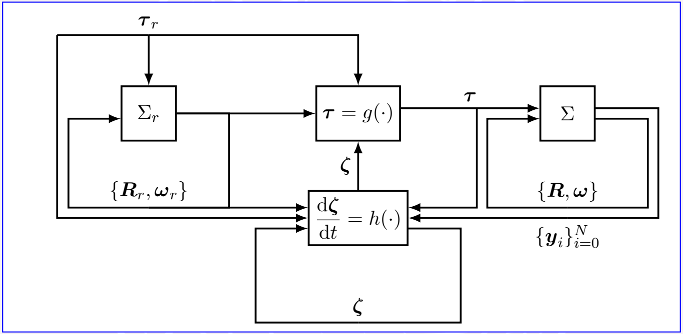

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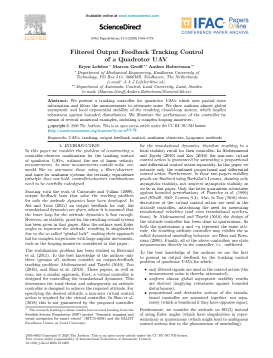
- Define an update of ζ in a set of measurements $\{\boldsymbol{y}_i\}_{i=1}^N$ (IMU).
- Define a feedback law $g(\zeta, \boldsymbol{R}_r, \boldsymbol{\omega}_r, \boldsymbol{\tau}_r)$ such that $\boldsymbol{R} \to \boldsymbol{R}_r, \boldsymbol{\omega} \to \boldsymbol{\omega}_r$ and $\{\zeta, \boldsymbol{R}, \boldsymbol{\omega}, \boldsymbol{\tau}\}$ remain bounded.

Introduction

Case: *filtered output feedback* in [1].

- Illustrate the ideas
- Less focus on precision
- Defining the errors
- A curious Lyapunov function
- Applying Barbălat
- Applying Matrosov
- Simulation example



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[1] E. Lefeber, M. Greiff, and A. Robertsson, “Filtered output feedback tracking control of a quadrotor UAV,” *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 5764–5770, 2020

Case study - Errors

Consider the errors,

$$\boldsymbol{R}_e = \boldsymbol{R}_r \boldsymbol{R}^\top \in \mathrm{SO}(3), \tag{7a}$$

$$\tilde{\boldsymbol{R}} = \hat{\boldsymbol{R}} \boldsymbol{R}^\top \in \mathrm{SO}(3), \tag{7b}$$

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_r - \boldsymbol{\omega} \in \mathbb{R}^3, \tag{7c}$$

$$\tilde{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} - \boldsymbol{\omega} \in \mathbb{R}^3, \tag{7d}$$

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With the controller and observer in [1] (here omitted for brevity),

$$\dot{\boldsymbol{R}}_e = f_1(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \tag{8a}$$

$$\dot{\tilde{\boldsymbol{R}}}_e = f_2(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \tag{8b}$$

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_e = f_3(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \tag{8c}$$

$$\boldsymbol{J}\dot{\tilde{\boldsymbol{\omega}}} = f_4(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \tag{8d}$$

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Case Study - Lyapunov

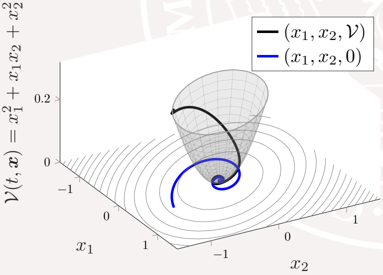
Consider an AS linear system,

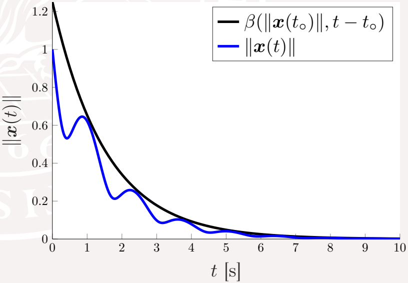
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t_\circ) \in \mathbb{R}^n.$$

Then, there exists a solution

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} = \boldsymbol{0}, \quad \boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{0}, \quad \boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{0},$$

using `lyap(A,Q)` and a quadratic Lyapunov function $\mathcal{V} = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$.





Case study - Lyapunov

Consider a PD function in the errors (where $k_i > 0$, $\boldsymbol{v}_i \nparallel \boldsymbol{v}_j \in \mathbb{R}^3$),

$$\mathcal{V}_1 = \sum_{i=1}^{N} \frac{k_i}{2} \|\boldsymbol{R}_e \tilde{\boldsymbol{R}}^\top \boldsymbol{v}_i - \boldsymbol{v}_i\|^2 + \frac{1}{2}\boldsymbol{\omega}_e^\top \boldsymbol{J} \boldsymbol{\omega}_e + \sum_{i=1}^{N} \frac{k_i}{2} \|\tilde{\boldsymbol{R}} \boldsymbol{v}_i - \boldsymbol{v}_i\|^2 + \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \boldsymbol{J} \tilde{\boldsymbol{\omega}}.$$

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The function is NSD along the solutions of the error dynamics, with

$$\dot{\mathcal{V}}_1 = -c_R \Big\| \sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i)(\boldsymbol{R}_r{}^\top \boldsymbol{v}_i + \boldsymbol{R}^\top \boldsymbol{v}_i) \Big\|^2 - \boldsymbol{\omega}_e^\top \boldsymbol{K}_\omega \boldsymbol{\omega}_e - \tilde{\boldsymbol{\omega}}^\top \boldsymbol{C}_\omega \tilde{\boldsymbol{\omega}}.$$

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(i) Standard Lyapunov theory is difficult to apply

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(ii) As $\dot{\mathcal{V}}_1$ is negative semi-definite, $\mathcal{V}_1$ is upper bounded,

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(iv) Due to (iii), $\dot{\mathcal{V}}_1$ is uniformly continuous in time

Case study - Barbălat

Lemma (Barbălat '59 [9])

Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *be a uniformly continuous function on its domain. If* $\Phi(t) = \lim_{t\to\infty} \int_0^t \phi(\tau) d\tau$ *exists and is finite,* $\phi(t) \to 0$ *as* $t \to \infty$.

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Application of the Lemma yields asymptotic convergence to

$$\lim_{(t-t_\circ)\to\infty} \dot{\mathcal{V}}_1 = \mathbf{0} \Rightarrow \begin{cases} \lim\limits_{(t-t_\circ)\to\infty} \boldsymbol{\omega}_e = \mathbf{0} \\ \lim\limits_{(t-t_\circ)\to\infty} \tilde{\boldsymbol{\omega}} = \mathbf{0} \\ \lim\limits_{(t-t_\circ)\to\infty} \sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i)(\boldsymbol{R}_r^\top \boldsymbol{v}_i + \boldsymbol{R}^\top \boldsymbol{v}_i) = \mathbf{0} \end{cases}$$

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Case study - Barbălat

Lemma (Variant of Barbălat's Lemma [10, Lemma 2.2.12])

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *be any differentiable function. If* $f(t)$ *converges to zero as* $t \to \infty$ *and its derivative satisfies*

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \geq 0, \tag{9}$$

where $f_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is uniformly continuous and* $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$*. If* $\eta(t)$ *tends to zero as* $t \to \infty$*,* $\dot{f}(t)$ *and* $f_0(t)$ *tend to zero as* $t \to \infty$*.*

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where $f_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous and $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If $\eta(t)$ tends to zero as $t \to \infty$, $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.

Consider

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_e = f_3(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}})$$

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where $f_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous and $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If $\eta(t)$ tends to zero as $t \to \infty$, $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.

Consider

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_e = \boldsymbol{S}(\boldsymbol{J}\boldsymbol{\omega})\boldsymbol{\omega}_e + \boldsymbol{S}(\boldsymbol{J}\tilde{\boldsymbol{\omega}})\boldsymbol{\omega}_r - \boldsymbol{K}_\omega(\boldsymbol{\omega}_e - \tilde{\boldsymbol{\omega}}) - \sum_{i=1}^{N} k_i \boldsymbol{S}(\boldsymbol{R}_r^\top \boldsymbol{v}_i)\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i,$$

Case study - Barbălat

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As $f(t) \to 0$ and $f_0(t)$ is uniformly continuous, $f_0(t) \to 0$ as $t \to \infty$.

Case study - Barbălat

Summary from Barbălat

- First application,

$$\sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i)(\boldsymbol{R}_r^\top \boldsymbol{v}_i + \boldsymbol{R}^\top \boldsymbol{v}_i) \triangleq f_0(t) + g_0(t) \to \mathbf{0}.$$

- Signal chasing, $f_0(t) \to 0 \Rightarrow g_0(t) \to 0$ as $(t - t_\circ) \to \infty$

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All solutions converge to an invariant set

$$\mathcal{S} = \left\{ (\boldsymbol{R}_e, \tilde{\boldsymbol{R}}, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \in \mathrm{SO}(3)^2 \times \mathbb{R}^6 \;\middle|\; \begin{matrix} \sum_{i=1}^{N} k_i \boldsymbol{S}(\boldsymbol{R}_r^\top \boldsymbol{v}_i)\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i = \boldsymbol{0} \\ \sum_{i=1}^{N} k_i \boldsymbol{S}(\boldsymbol{R}^\top \boldsymbol{v}_i)\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i = \boldsymbol{0} \\ \boldsymbol{\omega}_e = \boldsymbol{0} \\ \tilde{\boldsymbol{\omega}} = \boldsymbol{0} \end{matrix} \right\}.$$

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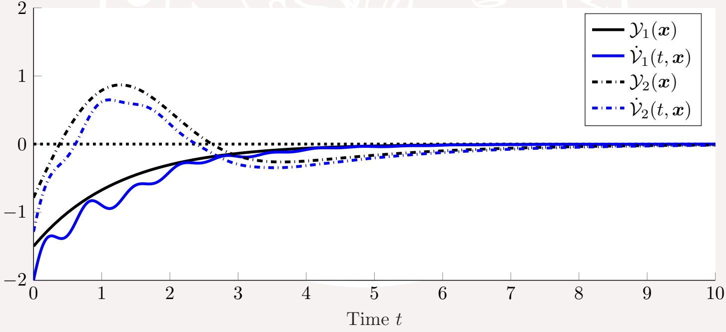
However, convergence to $\mathcal{S}$ is asymptotic, but not necessarily *uniform*.

Case study - Matrosov

The main idea of Matrosov

- NSD $\dot{\mathcal{V}}_1$ and non-autonomous error dynamics
- Find uniformly bounded function $\mathcal{Y}_i$ which upper bounds $\dot{\mathcal{V}}_i$
- Satisfy nested properties on $\mathcal{Y}_i$
- Exact details in [3, Thm. 1 and Thm. 2].



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In this context, let

- $\dot{\mathcal{V}}_1 \triangleq \mathcal{Y}_1$
- $\mathcal{V}_2 = \boldsymbol{\omega}_e^\top \sum_{i=1}^N k_i \boldsymbol{S}(\boldsymbol{R}_r^\top \boldsymbol{v}_i)\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i.$

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- Then, plugging in the error dynamics,

$$\frac{\mathrm{d}\mathcal{V}_2}{\mathrm{d}t} \leq -\left\|\sum_{i=1}^N k_i \boldsymbol{S}(\boldsymbol{R}_r^\top \boldsymbol{v}_i)\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i\right\|^2 + M_3 \left\|\begin{bmatrix}\boldsymbol{\omega}_e\\ \tilde{\boldsymbol{\omega}}\end{bmatrix}\right\| + M_4 \left\|\begin{bmatrix}\boldsymbol{\omega}_e\\ \tilde{\boldsymbol{\omega}}\end{bmatrix}\right\|^2 \triangleq \mathcal{Y}_2,$$

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- $\mathcal{Y}_1 = 0 \Rightarrow \boldsymbol{\omega}_e = \tilde{\boldsymbol{\omega}} = \mathbf{0} \Rightarrow \mathcal{Y}_2 \leq 0$, and

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- $\mathcal{Y}_1 = 0 \Rightarrow \boldsymbol{\omega}_e = \tilde{\boldsymbol{\omega}} = \boldsymbol{0} \Rightarrow \mathcal{Y}_2 \leq 0$, and
- $\mathcal{Y}_1 = \mathcal{Y}_2 = 0 \Rightarrow (\boldsymbol{R}_e, \tilde{\boldsymbol{R}}, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \rightarrow \mathcal{S}$.

Case study - Matrosov

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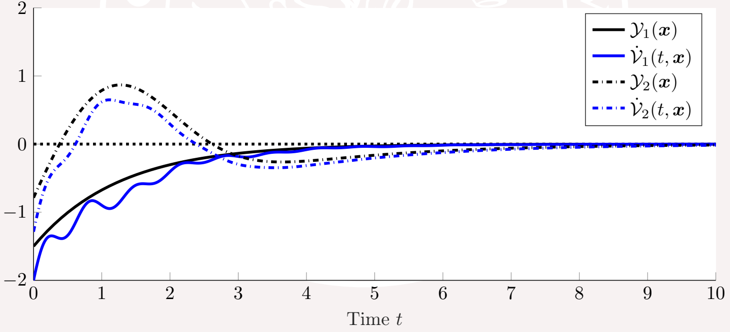


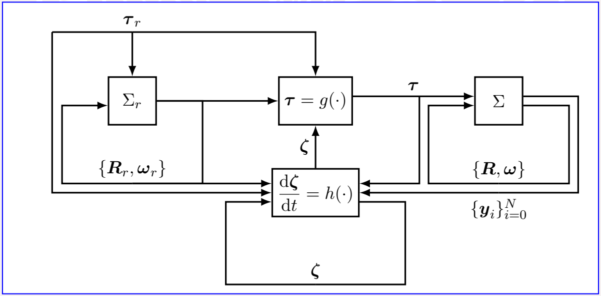
Case study - Summary

Barbalat + Signal chasing + Matrosov

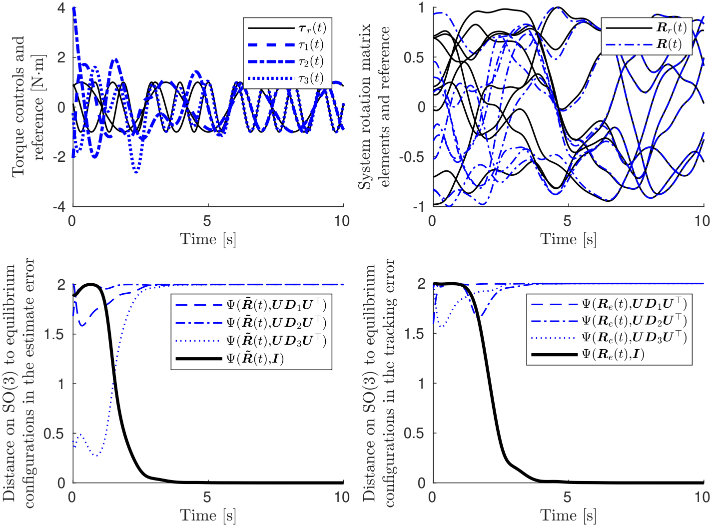
- Uniform asymptotic convergence of $(\boldsymbol{R}_e, \tilde{\boldsymbol{R}}, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \to \mathcal{S}$

It is also possible to show

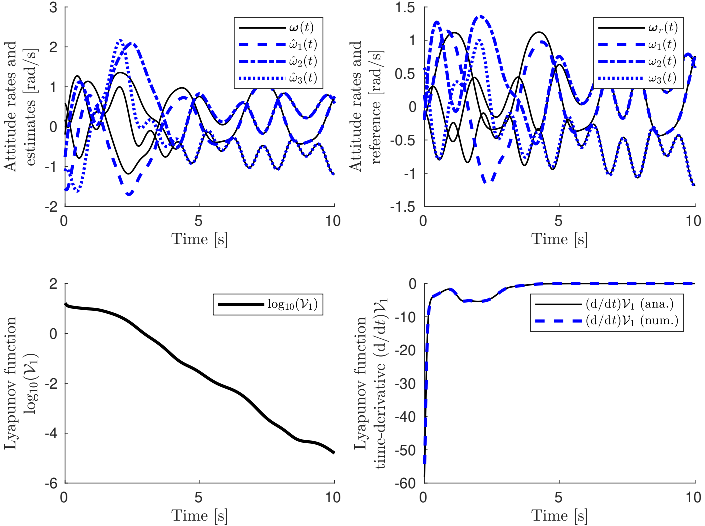
- By conditions on $\{(k_i, \boldsymbol{v}_i)\}_{i=1}^N$: $\mathcal{S}$ contains 13 isolated equilibrium points
- By local linearization : $(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}) \in \mathcal{S}$ is UAGAS
- By local linearization : $(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}) \in \mathcal{S}$ is ULES



Case study - Simulation



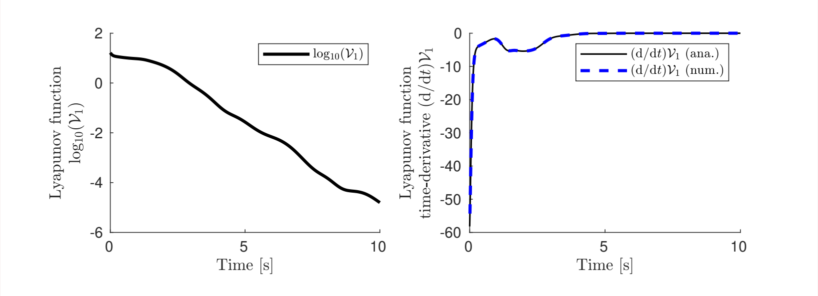
Case study - Simulation



Case study - Simulation

Recall, the proposed Lyapunov function

$$\mathcal{V}_1 = \sum_{i=1}^{N} \frac{k_i}{2} \|\boldsymbol{R}_e \tilde{\boldsymbol{R}}^\top \boldsymbol{v}_i - \boldsymbol{v}_i\|^2 + \frac{1}{2}\boldsymbol{\omega}_e^\top \boldsymbol{J} \boldsymbol{\omega}_e + \sum_{i=1}^{N} \frac{k_i}{2} \|\tilde{\boldsymbol{R}} \boldsymbol{v}_i - \boldsymbol{v}_i\|^2 + \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \boldsymbol{J} \tilde{\boldsymbol{\omega}}.$$

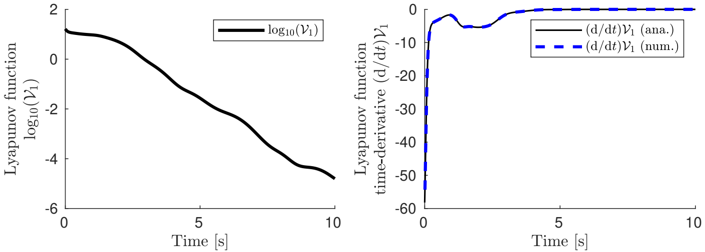


Case study - Simulation

- The Lyapunov function time derivative in the errors (black)

$$\dot{\mathcal{V}}_1 = -c_R \Big\| \sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i)(\boldsymbol{R}_r{}^\top \boldsymbol{v}_i + \boldsymbol{R}^\top \boldsymbol{v}_i) \Big\|^2 - \boldsymbol{\omega}_e^\top \boldsymbol{K}_\omega \boldsymbol{\omega}_e - \tilde{\boldsymbol{\omega}}^\top \boldsymbol{C}_\omega \tilde{\boldsymbol{\omega}}.$$

- And evaluated from $\mathcal{V}_1$ by numerical differentiation (blue)



Conclusions

Summary

- Separation principle and peaking
- Uniform stability and robustness
- Tools from Lyapunov, Barbalat, and Matrosov
- Making sense of 27 error signals
- Code: AerialVehicleControl.jl [11]
- More: ACC Wed, 10.15 and 11.00 (UTC -5)

Thank you for listening!

[11] M. Greiff, *AerialVehicleControl.jl, nonlinear and robust UAV control system synthesis*, github.com/mgreiff/AerialVehicleControl.jl, 2020

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