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Filtered Output Feedback Tracking Control of a Quadrotor UAV

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Abstract: We present a tracking controller for quadrotor UAVs which uses partial state information and filters the measurements to attenuate noise. We show uniform almost global asymptotic and local exponential stability of the resulting closed-loop system, which implies robustness against bounded disturbances. We illustrate the performance of the controller by means of several numerical examples, including a complex looping maneuver.

Keywords: UAVs, tracking, output feedback control, nonlinear observers, Lyapunov methods

1. INTRODUCTION

In this paper we consider the problem of constructing a controller-observer combination for the tracking control of quadrotor UAVs, without the use of linear velocity measurements. As state measurements contain noise, one would like to attenuate those using a filter/observer, and since for nonlinear systems the certainty equivalence principle does not hold, controller-observer combinations need to be carefully codesigned.

Starting with the work of Caccavale and Villani (1999), output feedback laws that solve the tracking problem for only the attitude *dynamics* have been developed. In Asl and Yoon (2015) an output feedback for only the translational dynamics are given, where it is assumed that the inner loop for the attitude dynamics is fast enough. However, no stability proof for the resulting overall system has been given in that paper. Also, the authors used Euler angles to represent the attitude, resulting in singularities due to the so called "gimbal lock", making their approach fail for complex trajectories with large angular movements, such as the looping maneuver considered in this paper.

The stabilization problem has been studied in Bertrand et al. (2011). To the best knowledge of the authors only three (groups of) authors consider an output-feedback tracking problem: Abdessameud and Tayebi (2010), Zou (2016), and Shao et al. (2018). Those papers, as well as ours, use a similar approach. First, a virtual controller is designed for controlling the translational dynamics. This determines the total thrust and subsequently an attitude controller is designed to achieve the required attitude. For specifying the desired attitude, a non-zero virtual control action is required for the virtual controller. In Shao et al. (2018) this is not guaranteed by the proposed controller for the translational dynamics, therefore resulting in a local stability result for their controller. In Abdessameud and Tayebi (2010) and Zou (2016) the non-zero virtual control action is guaranteed by saturating a proportional and differential control action separately. In this paper we saturate only the combined proportional and differential control action. Furthermore, in those two papers stability proofs are finalized using Barbălat's Lemma, showing only asymptotic stability, not *uniform* asymptotic stability as we do in this paper. Only the latter guarantees robustness against bounded perturbations, cf. Panteley et al. (1999) and (Khalil, 2002, Lemma 9.3). Also, in Zou (2016) time-derivatives of the virtual control action are used in the attitude controller, introducing the need for measuring translational velocities (and even translational accelerations). In Abdessameud and Tayebi (2010) the design of the attitude controller has been done in quaternions. As both the quaternions q and $-q$ represent the same attitude, the resulting attitude controller may exhibit the so called dynamical unwinding behavior, see Bhat and Bernstein (2000). Finally, all of the above controllers use state measurements directly in the controller, i.e., unfiltered.

To the best knowledge of the authors we are the first to present an output feedback for the tracking control problem of quadrotor UAVs for which:

- only filtered signals are used in the control action (the measurement noise is thereby attenuated),
- *uniform* almost global asymptotic stability results are derived (implying robustness against bounded disturbances),
- proportional and derivative actions of the translational controller are saturated together, not separately (which is beneficial if they have opposite signs).

Furthermore, we consider the attitude on SO(3) instead of using Euler angles (which have singularities in representation) or quaternions (which might lead to ambiguous control actions due to the phenomenon of unwinding).

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This paper is outlined as follows. In Section 2 we introduce some notation and preliminaries that are used throughout the paper. The problem formulation is presented in Section 3, after which a virtual filtered controller for the translational dynamics is presented in Section 4. A filtered controller for the attitude dynamics is presented in Section 5, after which stability of the combined result is shown in Section 6. The theoretical results are illustrated by a set of simulation examples in Section 7, where the propositions in each of the above sections are demonstrated separately. A final example is given with filtered output feedback control of a UAV in a looping manoeuvre on the surface of a torus, and Section 8 finally closes the paper. An extended version of the paper can be found in Lefeber et al. (2020).

2. PRELIMINARIES

In this section we introduce the notation, definitions and theorems used in the remainder of this paper.

Let e_i for $i \in \{1,2,3\}$ denote the standard unit vector, and let x_i denote the i^{th} element of a vector x. For definitions of uniform global (or local) asymptotic (or exponential) stability (UGAS/UGES/ULES), refer to Khalil (2002).

Definition 1. The origin of (2) is *uniformly almost globally asymptotically stable (UaGAS)* if it is UGAS, except for initial conditions in a set of measure zero.

We consider rotations $R \in \mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^\top R = I, \det R = 1\}$, and define the skew-symmetric map

$$S(a) = -S(a)^\top = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3). \qquad (1)$$

To compare elements of SO(3), we define a measure by its associated logarithmic map $\log : \mathrm{SO}(3) \to \mathfrak{so}(3)$, as

$$d(R_1, R_2) = \| \log(R_1 R_2^\top) \| \in [0, \pi].$$

Using the fact that the cross product $a \times b = S(a)b$ we have the following useful properties for the map S:

$$\begin{aligned} x^\top S(a) x &= 0 && \forall a, x \in \mathbb{R}^3, \\ S(a)b &= -S(b)a && \forall a, b \in \mathbb{R}^3, \\ RS(a) &= S(Ra)R && \forall R \in \mathrm{SO}(3), \forall a \in \mathbb{R}^3, \\ a^\top S(b)c = c^\top S(a) b &= b^\top S(c) a && \forall a, b, c \in \mathbb{R}^3. \end{aligned}$$

In the remainder, let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ denote a vector-function $\sigma(x) = s(\frac{1}{2}x^\top x)x$, where $s : \mathbb{R}^+ \to \mathbb{R}^+$ is a twice continuously differentiable function satisfying $s(0) > 0$ and for which the associated Lyapunov function

$$V_\sigma(x) = \int_0^{\frac{1}{2}x^\top x} s(\tau)\mathrm{d}\tau,$$

is positive definite and radially unbounded. Possible candidates are $\sigma(x) = k_0 x$ and $\sigma(x) = (k_\infty^2 + k_0^2 x^\top x)^{-1/2} k_0 k_\infty x$ with $k_0 > 0$ and $k_\infty > 0$, where the latter is bounded.

Definition 2. A function σ as considered above for which $\|\sigma(x)\| \leq \gamma$ for all $x \in \mathbb{R}^n$ is called a *saturation function.*

Theorem 1. (Corollary of Loría et al. (2005, Theorem 1)). Consider the dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad f(t, 0) = 0, \qquad (2)$$

with $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ locally bounded, continuous and locally uniformly continuous in t.

If there exist j differentiable functions $V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, bounded in t, and continuous functions $Y_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, 2, \ldots j\}$ such that

- V_1 is positive definite and radially unbounded,
- $\dot{V}_i(t, x) \leq Y_i(x)$, for all $i \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for $i \in \{1, 2, \ldots, k-1\}$ implies $Y_k(x) \leq 0$, for all $k \in \{1, 2, \ldots, j\}$,
- $Y_i(x) = 0$ for all $i \in \{1, 2, \ldots, j\}$ implies $x = 0$,

then the origin $x = 0$ of (2) is uniformly globally asymptotically stable (UGAS).

Theorem 2. (cf. Panteley and Loría (1998)). Let the system (2) be written as

$$\dot{x}_1 = f_1(t, x_1) + g(t, x_1, x_2)x_2 \qquad (3a)$$
$$\dot{x}_2 = f_2(t, x_2), \qquad (3b)$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $f_1(t, x_1)$ is continuously differentiable in (t, x_1) and $f_2(t, x_2)$, $g(t, x_1, x_2)$ are continuous in their arguments, and locally Lipschitz in x_2 and (x_1, x_2) respectively. This system is a cascade of the systems

$$\dot{x}_1 = f_1(t, x_1), \qquad (4)$$

and (3b). If the origins of the systems (4) and (3b) are UGAS and solutions of (3) remain bounded, then the origin of the system (3) is UGAS. In addition, if the systems (4) and (3b) are ULES, then (3) is also ULES.

Lemma 1. (cf. Mahony et al. (2008)). Let $k_i > 0$ and $v_i \in \mathbb{R}^3$ be such that $M = \sum_{i=1}^n k_i v_i v_i^\top = U \Lambda U^\top$ with Λ a diagonal matrix with distinct eigenvalues λ_i where $U \in \mathrm{SO}(3)$. Then $\sum_{i=1}^n k_i S(v_i) R v_i = 0$ implies that $U^\top R U \in \{I, D_1, D_2, D_3\}$, where $D_1 = \mathrm{diag}(1, -1, -1)$, $D_2 = \mathrm{diag}(-1, 1, -1)$, $D_3 = \mathrm{diag}(-1, -1, 1)$.

3. PROBLEM FORMULATION

Let $\rho \in \mathbb{R}^3$ denote the position of the centre of mass relative to a North-East-Down (NED) inertial frame. Let $R \in \mathrm{SO}(3)$ denote the rotation matrix from the body-fixed frame to the inertial frame. Furthermore, let $\nu \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$ denote the body-fixed linear and angular velocities. In this context, the SE(3)-configured UAV dynamics (comprehensively derived in Lee et al. (2017)) can be written

$$\dot{\rho} = R\nu \qquad (5a)$$
$$\dot{\nu} = -S(\omega)\nu + gR^\top e_3 - (f/m)e_3 \qquad (5b)$$
$$\dot{R} = RS(\omega) \qquad (5c)$$
$$J\dot{\omega} = S(J\omega)\omega + \tau, \qquad (5d)$$

where m denotes the total mass, $J = J^\top > 0$ the inertia matrix with respect to the body-fixed frame, the matrix S is given by (1), and $f \in \mathbb{R}$ and $\tau \in \mathbb{R}^3$ denote respectively the total thrust magnitude and the total moment vector in the body-fixed frame, which are assumed to be the inputs.

Assume that we are given a feasible continuous reference trajectory $(\rho_r, R_r, \nu_r, \omega_r, \tau_r, f_r, \dot{f}_r, \ddot{f}_r)$, satisfying

$$\dot{\rho}_r = R_r \nu_r \qquad (6a)$$
$$\dot{\nu}_r = -S(\omega_r)\nu_r + gR_r^\top e_3 - (f_r/m)e_3 \qquad (6b)$$
$$\dot{R}_r = R_r S(\omega_r) \qquad (6c)$$
$$J\dot{\omega}_r = S(J\omega_r)\omega_r + \tau_r, \qquad (6d)$$

where $0 < f_r^{\min} \leq f_r(t)$.

Define the following error coordinates on SE(3):

$$\bar\rho = R_r^\top(\rho-\rho_r) \qquad \bar R = R_r^\top R$$
$$\bar\nu = -\bar R^\top S(\omega_r)\bar\rho + \nu - \bar R^\top \nu_r \qquad \bar\omega = \omega - \bar R^\top\omega_r$$

with corresponding error measure:

$$\varepsilon(\bar\rho,\bar R,\bar\nu,\bar\omega) = \|\bar\rho\| + \|\log\bar R\| + \|\bar\nu\| + \|\bar\omega\|.$$

Then we can define the filtered tracking control problem.

Problem 1. For $(\rho_r, R_r, \nu_r, \omega_r, \tau_r, f_r, \dot f_r, \ddot f_r)$ being a given feasible reference trajectory, find appropriate control laws

$$f = f(\zeta,\rho_r,R_r,\nu_r,\omega_r) \tag{7a}$$
$$\tau = \tau(\zeta,\rho_r,R_r,\nu_r,\omega_r) \tag{7b}$$
$$\dot\zeta = \zeta(\rho,R,\omega,z,\rho_r,R_r,\nu_r,\omega_r), \tag{7c}$$

where ζ denotes the memory of the filter, such that for the resulting closed-loop system (5), (6), (7)

$$\lim_{t\to\infty}\varepsilon\big(\bar\rho(t),\bar R(t),\bar\nu(t),\bar\omega(t)\big) = 0.$$

4. FILTERED POSITION TRACKING CONTROL

Following Lefeber et al. (2017) we separate the design of the tracking controller into two parts. In this section we consider the derivation of a position tracking controller under the assumption that we can use the body-fixed linear accelerations as (virtual) input. In subsequent sections we consider the problem of realizing this virtual input by means of the actual inputs. First, we define the tracking error in the body-fixed frame of the reference:

$$\begin{bmatrix}\rho_e\\ \nu_e\end{bmatrix} = \begin{bmatrix}R_r^\top(\rho_r-\rho)\\ \nu_r - R_r^\top R\nu\end{bmatrix}.$$

Using this definition the tracking error dynamics become

$$\dot\rho_e = -S(\omega_r)\rho_e + \nu_e$$
$$\dot\nu_e = -S(\omega_r)\nu_e + (f/m)R_r^\top R e_3 - (f_r/m)e_3.$$

For stabilizing these time-varying tracking error dynamics we take $u = (f/m)R_r^\top Re_3 - (f_r/m)e_3$ to be a virtual input which we want to achieve by controlling the thrust magnitude and the attitude, leading to the first proposition.

Proposition 1. The dynamics

$$\dot\rho_e = -S(\omega_r)\rho_e + \nu_e \tag{8a}$$
$$\dot\nu_e = -S(\omega_r)\nu_e + u, \tag{8b}$$

in closed-loop with the dynamic output feedback

$$u = -\sigma(k_\rho\hat\rho_e + k_\nu\hat\nu_e) \tag{9a}$$
$$\dot{\hat\rho}_e = -S(\omega_r)\hat\rho_e + \hat\nu_e + L_1 z \tag{9b}$$
$$\dot{\hat\nu}_e = -S(\omega_r)\hat\nu_e + u + L_2 z \tag{9c}$$
$$\dot z = -S(\omega_r)z - (L_1+L_3)z + (L_1+L_3)\tilde\rho_e, \tag{9d}$$

with $k_\rho > 0$ and $k_\nu > 0$, $L_1 > 0$, $L_2 > 0$, and $L_3 > 2L_2/L_1$ is UGAS and ULES.

Proof. Define the errors $\tilde\rho_e = \rho_e - \hat\rho_e$, $\tilde\nu_e = \nu_e - \hat\nu_e$, $\tilde z = z - \tilde\rho_e$, and also $\hat e = k_\rho\hat\rho_e + k_\nu\hat\nu_e$. Then we obtain

$$\dot{\hat e} = -S(\omega_r)\hat e + k_\rho\hat\nu_e + k_\nu u + (k_\rho L_1 + k_\nu L_2)z \tag{10a}$$
$$\dot{\tilde\rho}_e = -S(\omega_r)\tilde\rho_e + \tilde\nu_e - L_1 z \tag{10b}$$
$$\dot{\tilde\nu}_e = -S(\omega_r)\tilde\nu_e - L_2 z \tag{10c}$$
$$\dot{\tilde z} = -S(\omega_r)\tilde z - L_3\tilde z + L_1\tilde\rho_e - \tilde\nu_e, \tag{10d}$$

Consider the Lyapunov function candidate

$$V_1(\rho_e,\nu_e,\tilde\rho_e,\tilde\nu_e,\tilde z) = V_\sigma(\hat e) + \frac12 k_\rho\nu_e^\top\nu_e + {} + \frac{\alpha}{2}(\tilde\rho_e - \beta\tilde\nu_e)^\top(\tilde\rho_e-\beta\tilde\nu_e) + \frac{\alpha\gamma}{2}\tilde\nu_e^\top\tilde\nu_e + \frac{\alpha}{6}\tilde z^\top\tilde z, \tag{11}$$

with $\beta = \frac{2L_1}{3L_2}$, $\gamma = \frac{2L_1^2}{9L_2^2} + \frac{1}{L_2}$, and α sufficiently large:

$$\alpha > \frac{\max\big((k_\rho L_1 + k_\nu L_2)^2, k_\rho^2\big)}{k_\nu\min\left(\frac13 L_1, \frac{2L_1+L_2L_3-\sqrt{4L_1^2+16L_2^2-4L_1L_2L_3+L_2^2L_3^2}}{6L_2}\right)}.$$

Differentiating (11) along (8), (9), (10) results in

$$\dot V_1(\rho_e,\nu_e,\tilde\rho_e,\tilde\nu_e,\tilde z) = -k_\nu\sigma(\hat e)^\top\sigma(\hat e) - k_\rho\sigma(\hat e)^\top\tilde\nu_e^\top + {} + (k_\rho L_1 + k_\nu L_2)\sigma(\hat e)^\top\tilde\rho_e + (k_\rho L_1+k_\nu L_2)\sigma(\hat e)^\top\tilde z - \alpha\Big[\frac13 L_1\tilde\rho_e^\top\tilde\rho_e + \frac{2L_1}{3L_2}\tilde\nu_e^\top\tilde\nu_e + \frac43\tilde\nu_e^\top\tilde z + \frac13 L_3\tilde z^\top\tilde z\Big], \tag{12}$$

which is negative semi-definite function in its arguments, but notably negative definite in $\sigma(\hat e)$, $\tilde\rho_e$, $\tilde\nu_e$, and $\tilde z$.

Differentiating $V_2 = -\tilde\nu_e^\top z$ along (8), (9), (10) results in

$$\dot V_2 = -L_2 z^\top z - \sigma(\hat e)^\top z + (L_1+L_3)\tilde\nu_e^\top(z-\tilde\rho_e) = Y_2.$$

Differentiating $V_3 = -\hat e^\top\hat\nu_e$ along (8), (9), (10) yields

$$\dot V_3 = -k_\rho\hat\nu_e^\top\hat\nu_e - \sigma(\hat e)^\top(k_\nu\hat\nu_e + \hat e) - {} - z^\top[(k_\rho L_1 + k_\nu L_2)\hat\nu_e + L_2\hat e] = Y_3.$$

Applying Theorem 1 completes the proof of UGAS. ULES follows from a linearization at the stable equilibrium. □

Remark 1. Here, it is useful to note that

$$\frac{\mathrm d}{\mathrm dt}\sigma(x) = \frac{\mathrm d}{\mathrm dt}s(\tfrac12 x^\top x)x = s'(\tfrac12 x^\top x)x^\top\dot x x + s(\tfrac12 x^\top x)\dot x$$
$$\frac{\mathrm d^2}{\mathrm dt^2}\sigma(x) = s''(\tfrac12 x^\top x)(x^\top\dot x)^2 x + s'(\tfrac12 x^\top x)\big[\dot x^\top\dot x x + x^\top\ddot x x + {} + 2x^\top\dot x\dot x\big] + s(\tfrac12 x^\top x)\ddot x,$$

and

$$\ddot{\hat e} = -S(\dot\omega_r)\hat e - S(\omega_r)\dot{\hat e} + k_\rho\dot{\hat\nu}_e + k_\nu\frac{\mathrm d\sigma(\hat e)}{\mathrm dt} + (k_\rho L_1 + k_\nu L_2)\dot z.$$

Therefore, $\dot u$ and $\ddot u$ can be expressed as continuous functions of signals that are available from measurements.

5. FILTERED ATTITUDE CONTROL

In Section 6 we want to achieve the input derived in the previous section by means of filtered attitude control, but before we can do so, we first need to construct a filtered attitude controller for tracking reference dynamics.

Proposition 2. Consider the dynamics

$$\dot R = RS(\omega) \qquad \dot R_r = R_r S(\omega_r) \tag{13a}$$
$$J\dot\omega = S(J\omega)\omega + \tau \qquad J\dot\omega_r = S(J\omega_r)\omega_r + \tau_r. \tag{13b}$$

Define the errors $R_e = R_r R^\top$, $\tilde R = \hat R R^\top$, $\omega_e = \omega_r - \omega$, and $\tilde\omega = \hat\omega - \omega$, and let $\hat\omega_e = \omega_r - \hat\omega$. Then the input

$$\tau = \tau_r + S(J\hat\omega_e)\omega_r + K_\omega\hat\omega_e + \sum_{i=1}^n k_i S(R_r^\top v_i)\hat R^\top v_i \tag{14a}$$
$$\dot{\hat R} = \hat R S(\omega + \delta_R) \tag{14b}$$
$$J\dot{\hat\omega} = S(J\omega)\omega + \tau + \delta_\omega, \tag{14c}$$

where the innovation terms δ_R and δ_ω are given by

$$\delta_R = -c_R \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)(R_r^\top v_i + R^\top v_i) \tag{14d}$$

$$\delta_\omega = -c_\omega J S(\omega_r)\omega_e - c_\omega K_\omega \omega_e - C_\omega \tilde{\omega}, \tag{14e}$$

with $K_\omega = K_\omega^\top > 0$, $C_\omega = C_\omega^\top > 0$, $c_R > 0$, $c_\omega > 0$, and $k_i > 0$ such that $M = \sum_{i=1}^n k_i v_i v_i^\top$ has distinct eigenvalues, renders the equilibrium point $(R_e, \tilde{R}, \omega_e, \tilde{\omega}) = (I, I, 0, 0)$ UaGAS and ULES. That is, let $E_c = \{I, UD_1U^\top, UD_2U^\top, UD_3U^\top\}$ with $U \in \mathrm{SO}(3)$ such that $M = U\Lambda U^\top$ with Λ being a diagonal matrix. Then R_e and $\tilde{R}$ converge to E_c, and ω_e and $\tilde{\omega}$ converge to zero. The equilibria where $R_e \in E_c \setminus \{I\}$ or $\tilde{R} \in E_c \setminus \{I\}$ are unstable and the set of all initial conditions converging to these equilibria form a lower-dimensional manifold.

Proof. The closed-loop dynamics (13), (14) is given by

$$\dot{R}_e = S(R_r\omega_e)R_e \tag{15a}$$

$$J\dot{\omega}_e = S(J\omega)\omega_e + S(J\tilde{\omega})\omega_r - K_\omega\tilde{\omega}_e - \sum_{i=1}^{n} k_i S(R_r^\top v_i)\hat{R}^\top v_i$$

$$\dot{\tilde{R}} = S(\hat{R}\delta_R)\tilde{R} \tag{15b}$$

$$J\dot{\tilde{\omega}} = \delta_\omega \tag{15c}$$

Differentiating the Lyapunov function candidate

$$V_1 = \sum_{i=1}^{n} \frac{k_i}{2}(R_e\tilde{R}^\top v_i - v_i)^\top(R_e\tilde{R}^\top v_i - v_i) + \frac{1}{2}\omega_e^\top J\omega_e + \sum_{i=1}^{n} \frac{k_i}{2}(\tilde{R}v_i - v_i)^\top(\tilde{R}v_i - v_i) + \frac{1}{2c_\omega}\tilde{\omega}^\top J\tilde{\omega},$$

along solutions of (15) results in

$$\begin{aligned}\dot{V}_1 &= -[\omega_e - \delta_R]^\top \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)R_r^\top v_i + \omega_e^\top J\dot{\omega}_e \\ &\quad + \delta_R^\top \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)R^\top v_i + \frac{1}{c_\omega}\tilde{\omega}^\top J\dot{\tilde{\omega}} \\ &= -c_R \left\| \sum_{i=1}^{n} k_i S(\hat{R}^\top v_i)(R_r^\top v_i + R^\top v_i) \right\|_2^2 - \omega_e^\top K_\omega \omega_e - \tilde{\omega}^\top \frac{C_\omega}{c_\omega}\tilde{\omega},\end{aligned}$$

Differentiating $V_2 = \omega_e^\top \sum_{i=1}^n k_i S(R_r^\top v_i)\hat{R}^\top v_i$ along (15),

$$\dot{V}_2 \leq -\left\| \sum_{i=1}^{n} k_i S(R_r^\top v_i)\hat{R}^\top v_i \right\|_2^2 + M_1 \left\| \begin{bmatrix} \tilde{\omega}_e \\ \tilde{\omega} \end{bmatrix} \right\| + M_2 \left\| \begin{bmatrix} \tilde{\omega}_e \\ \tilde{\omega} \end{bmatrix} \right\|^2,$$

where we used boundedness of ω_r, and boundedness of $\tilde{\omega}$ and $\tilde{\omega}_e$ resulting from $\dot{V}_1 \leq 0$.

Applying Theorem 1 shows UGAS towards

$$\sum_{i=1}^{n} k_i S(R^\top v_i)\hat{R}^\top v_i = 0 \qquad \omega_e = 0$$

$$\sum_{i=1}^{n} k_i S(R_r^\top v_i)\hat{R}^\top v_i = 0 \qquad \tilde{\omega} = 0,$$

which along the lines of the proof in Mahony et al. (2008) implies UaGAS towards

$$R_e = I \qquad \omega_e = 0 \qquad \tilde{R} = I \qquad \tilde{\omega} = 0.$$

Considering $V_1 + \epsilon V_2$, ULES can be shown along the lines of the work of Wu and Lee (2016). □

6. COMBINED CONTROL/CASCADE ANALYSIS

In section 4 we derived a filtered controller for the translational dynamics by using a virtual input u. Subsequently, in section 5 we derived a filtered controller for the attitude dynamics. In this section we combine the results yielding a controller that solves Problem 1, using only filtered signals without the need for linear velocity measurements.

Following the approach in Lefeber et al. (2017), we need $fR_r^\top Re_3$ to converge to $mu + f_r e_3$. To this end, define

$$f_d = \begin{bmatrix} f_{d1} \\ f_{d2} \\ f_{d3} \end{bmatrix} = \frac{f_r e_3 + mu}{\|f_r e_3 + mu\|} \tag{16a}$$

as the desired thrust direction, satisfying $f_{d3} > 0$, provided that $\|u\| \leq f_r^{\min}/m$. We let

$$R_d = \begin{bmatrix} 1 - \frac{f_{d1}^2}{1+f_{d3}} & -\frac{f_{d1}f_{d2}}{1+f_{d3}} & f_{d1} \\ -\frac{f_{d1}f_{d2}}{1+f_{d3}} & 1 - \frac{f_{d2}^2}{1+f_{d3}} & f_{d2} \\ -f_{d1} & -f_{d2} & f_{d3} \end{bmatrix} \in \mathrm{SO}(3) \tag{16b}$$

denote the rotation matrix which rotates the desired thrust vector to the thrust vector of the reference (i.e., e_3) in the plane containing both vectors. This also gives

$$\omega_d = \begin{bmatrix} -\dot{f}_{d2} + \frac{f_{d2}\dot{f}_{d3}}{1+f_{d3}} \\ \dot{f}_{d1} - \frac{f_{d1}\dot{f}_{d3}}{1+f_{d3}} \\ \frac{f_{d2}\dot{f}_{d1} - f_{d1}\dot{f}_{d2}}{1+f_{d3}} \end{bmatrix}. \tag{16c}$$

Using $f = \|f_r e_3 + mu\|$ and (16b), we can write $f_r e_3 + mu = fR_d e_3$, so our goal to determine τ which makes $fR_r^\top Re_3$ converge to $f_r e_3 + mu$ can be replaced by the goal to determine τ which makes $R_r^\top R$ converge to R_d, or equivalently R to $R_r R_d$. The latter we can achieve by means of the filtered attitude controller of the previous section, where this time $R_e = R_r R_d R^\top$, and $\omega_e = R_d^\top \omega_r + \omega_d - \omega$, since we need R to converge to $R_r R_d$.

Proposition 3. Consider the dynamics (5) in closed-loop with the controller (14), $f = \|f_r e_3 + mu\|$, where $R_e = R_r R_d R^\top$, $\omega_e = R_d^\top \omega_r + \omega_d - \omega$, $\hat{\omega}_e = R_d^\top \omega_r + \omega_d - \hat{\omega}$, R_d and ω_d are given by (16) and u given by (9).

If σ is a saturation function satisfying $\|\sigma(x)\| \leq \gamma = f_r^{\min}/m$, $k_\rho > 0$, $k_\nu > 0$, $L_1 > 0$, $L_2 > 0$, $L_3 > 2L_2/L_1$, $K_\omega = K_\omega^\top > 0$, $C_\omega = C_\omega^\top > 0$, $c_R > 0$, $c_\omega > 0$, and $k_i > 0$ such that $M = \sum_{i=1}^n k_i v_i v_i^\top$ has distinct eigenvalues, then the equilibrium point $(\rho_e, \nu_e, \tilde{\rho}_e, \tilde{\nu}_e, \tilde{z}, R_e, \tilde{\omega}, \tilde{R}, \omega_e) = (0, 0, 0, 0, 0, I, 0, I, 0)$ is UaGAS and ULES. That is, let $E_c = \{I, UD_1U^\top, UD_2U^\top, UD_3U^\top\}$ with $U \in \mathrm{SO}(3)$ such that $M = U\Lambda U^\top$ with Λ being a diagonal matrix. Then R_e and $\tilde{R}$ converge to E_c and all other variables converge to zero. The equilibria where $R_e \in E_c \setminus \{I\}$ or $\tilde{R} \in E_c \setminus \{I\}$ are unstable and the set of all initial conditions converging to these equilibria form a lower dimensional manifold.

Proof. The resulting overall closed-loop dynamics can be written as (3) with

$$x_1 = [\rho_e,\ \nu_e,\ \tilde{\rho}_e,\ \tilde{\nu}_e,\ \tilde{z}]^\top$$

$$x_2 = \left[\sum_{i=1}^{n} k_i S(v_i) R_e v_i,\ \omega_e, \sum_{i=1}^{n} k_i S(v_i) \tilde{R} v_i,\ \tilde{\omega}\right]^\top$$

$$g(t, x_1, x_2)x_2 = [0\ 1\ 0\ 1\ 0]^\top \frac{\|f_r e_3 + mu\|}{m} R_r^\top (I - R_e) R e_3,$$

where $\dot{x}_1 = f(t, x_1)$ follows from (8), (9), (10) and is UGAS according to Proposition 1 and $\dot{x}_2 = f_2(t, x_2)$ follows from (15) which is UGAS and ULES by Proposition 2.

Differentiating V_1 as defined in (11) along (3a) results in

$$\dot{V}_1 = c_1 \sqrt{V} \|I - R_e\|$$

for some constant c_1. Since (3b) is ULES we have

$$\sqrt{V_1(t)} - \sqrt{V(t_0)} \leq c_2(x_2(t_0))$$

and therefore boundedness of solutions of (3). Applying Theorem 2 enables us to conclude that the cascaded system is UGAS and ULES. Therefore, the equilibrium point $(\rho_e, \nu_e, \tilde{\rho}_e, \tilde{\nu}_e, \tilde{z}, R_e, \tilde{\omega}, \tilde{R}, \omega_e) = (0, 0, 0, 0, 0, I, 0, I, 0)$ is UaGAS and ULES. □

Remark 1. The above mentioned controller also solves Problem 1, as $\rho_e \to 0$ implies $\bar{\rho} \to 0$, $R_e \to I$ implies $\bar{R} \to I$, and both together with $\nu_e \to 0$ and $\omega_e \to 0$ result in $\bar{\nu} \to 0$ and $\bar{\omega} \to 0$.

7. SIMULATION STUDIES

In this section, the stability and robustness of the presented controllers are illustrated in a set of simulations. We first demonstrate the filtered and saturated output feedback in Proposition 1, followed by an example of the attitude feedback in Proposition 2. Using these two propositions, the main result in Proposition 3 is demonstrated in a looping manoeuvre on the surface of a torus.

For the translational feedback, we define the saturation function in terms of the hyperbolic tangent function, with $\sigma(x) = \gamma \tanh(\|x\|_2/\gamma)\|x\|_2^{-1} x$ for some constant $\gamma > 0$.

Furthermore, in all subsequent examples, we use the initial conditions in Table 1 and parameter definitions in Table 2, where $\mathcal{N}(\mu, \Sigma)$ denotes a multivariate Gaussian distribution with mean μ and covariance Σ, and $\mathcal{U}(D)$ denotes a uniform distribution over a domain D. Any deviation from these parameters are stated explicitly in the examples.

Table 1. Initial conditions in the simulations.

Initial condition	Distribution	Description
$\rho(t_0), \hat{\rho}(t_0), z(t_0)$	$\mathcal{N}(0, I)$	Position (m)
$\nu(t_0), \hat{\nu}(t_0)$	$\mathcal{N}(0, I)$	Velocity (m/s)
$R(t_0), \hat{R}(t_0)$	$\mathcal{U}(SO(3))$	Attitude $(\cdot)$
$\omega(t_0), \hat{\omega}(t_0)$	$\mathcal{N}(0, I)$	Attitude rate (rad/s)

7.1 Saturated translational output feedback

In this first example, we consider the non-autonomous system in (8), for which the feedback loop is closed as described in Proposition 1. Here we only assume knowledge of the positional states, and take a time-varying $\omega_r(t)$, as

$$\dot{\omega}_r(t) = [\sin(t+1)\ \sin(2t+2)\ \sin(3t+3)]^\top.$$

We then generate three simulations from the same initial conditions, using the nominal parameters in Table 2 but changing the value of the saturation bound γ in each simulation. In the first, $\gamma = 1$ (red), in the second $\gamma = 1.5$ (blue), and in the third $\gamma = 2$ (black). The resulting system responses for these three cases are shown in Fig. 1, where the virtual input is clearly bounded at the corresponding γ-value at all times. The error states converge to zero, and the Lyapunov function (shown in the logarithm) decays exponentially in time, demonstrating the local exponential stability of the translational subsystem in Proposition 1.

Table 2. Parameters used in the simulations.

Parameter	Value	Description
(k_ρ, k_ν)	$(2, 2)$	Translational control gains
(L_1, L_2, L_3)	$(4, 4, 4)$	Translational filter gains
γ	2	Sat. bound ($\|u(t)\|_2 \leq \gamma$)
$(k_1, k_2, k_3, K_\omega)$	$(10, 20, 30, 15I)$	Attitude control gains
$(c_R, c_\omega, C_\omega)$	$(1, 10, 15I)$	Attitude filtering gains
v_1	$\begin{bmatrix}0 & 0 & -1\end{bmatrix}^\top$	Direction (gravity)
v_2	$\begin{bmatrix}0.98 & 0.17 & 0\end{bmatrix}^\top$	Direction (magnetic field)
v_3	$v_1 \times v_2$	Virtual meas. direction
J	$\frac{1}{100}\begin{bmatrix}5.2 & 2.2 & 2.2\\ 2.2 & 7.0 & 1.7\\ 2.2 & 1.7 & 5.3\end{bmatrix}$	Inertia tensor $(kg \cdot m^2)$
m	0.1	Mass (kg)
g	9.81	Gravitational acc. (m/s^2)

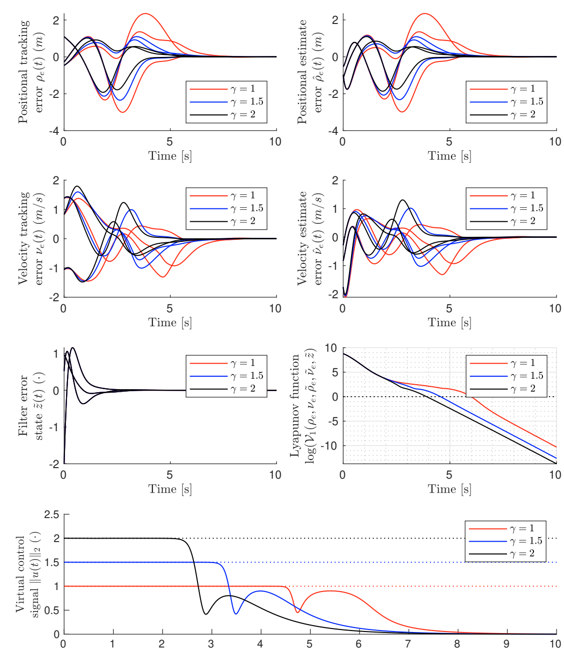

Fig. 1. From top to bottom: (i) tracking error and (ii) estimate error in position, (iii) tracking error and (iv) estimate error in velocity, (v) filter memory, (vi) Lyapunov function $V_1(\rho_e, \nu_e, \tilde{\rho}_e, \tilde{\nu}_e, \tilde{z})$ in the logarithm, and (vii) the norm of the virtual controls $\|u(t)\|_2$ for $\gamma = 1$ (red), $\gamma = 1.5$ (blue), $\gamma = 2$ (black).

7.2 Attitude output feedback

In this example, we illustrate the proposed filtered attitude output feedback in Proposition 2. The resulting closed loop system is UaGAS and ULES, and to show this in a simulation example, we consider a realization of the initial conditions and parameters in Tables 1 and 2, where the system is set to track a reference trajectory defined by the initial conditions $R_r(t_0) \sim \mathcal{U}(SO(3))$ and $\omega_r(t_0) \sim \mathcal{N}(0, I)$, and integrated in time with a reference torque

$$\tau_r(t) = [\sin(2t+1)\ \sin2(4t+2)\ \sin(6t+3)]\,. \qquad (17)$$

To illustrate the distance of two elements $R_1, R_2 \in SO(3)$, we consider the metric $d(R_1, R_2)$ defined in Section 2. With the resulting system response depicted in Fig. 2, it is clear that both the tracking error and the estimate error in the closed loop dynamics converge to the identity element, with $d(R_e, I) \to 0$ and $d(\tilde{R}, I) \to 0$. This despite an initialization which is close to as far from the stable equilibrium point as possible.

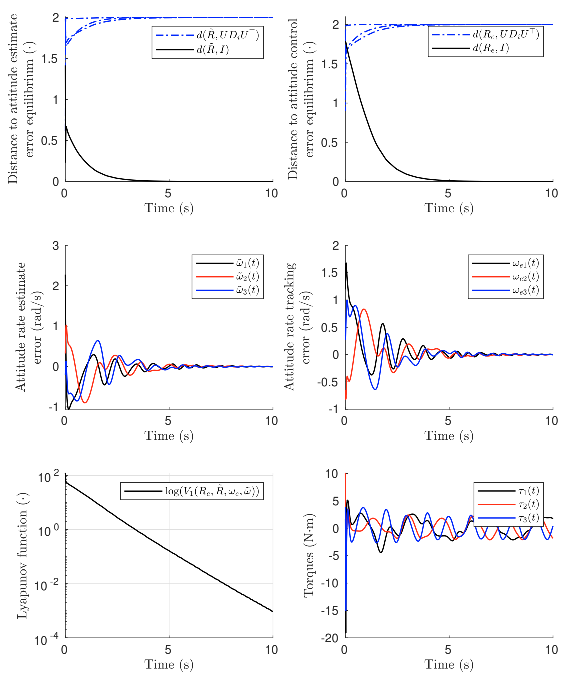

Fig. 2. From top left to bottom right; (i) attitude estimate error as $d(\tilde{R}, I)$; (ii) attitude tracking error as $d(R_e, I)$; (iii) the attitude rate estimate error; (iv) the attitude rate control error; (v) the Lyapunov function $V_1(R_e, \tilde{R}, \omega_e, \tilde{\omega})$ in the natural logarithm; (vi) the torque control signals $\tau(t)$; which clearly converge the defined reference trajectory $\tau_r(t)$ as defined in (17).

7.3 *Full output feedback with aggressive maneuvering*

We now consider the main result, combining the two output feedback controllers using the cascade theorem as outlined in Section 6. For this numerical example, we will attempt to track a highly volatile state-trajectory, defined as a looping maneuver on the surface of a torus. To facilitate such a demanding maneuver, we make use of the differential flatness of the quadrotor UAV, as derived in Greiff (2017) but here in the NED case. This permits the evaluation of a reference trajectory $(\rho_r, \nu_r, R_r, \omega_r, f_r, \dot{f}_r, \ddot{f}_r, \tau_r) \in \mathbb{R}^{15} \times SO(3)$ satisfying (6) from a set of flat outputs $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^\top \in \mathbb{R}^4$ without integration, provided the trajectory $\gamma(t)$ is sufficiently smooth. Refer to Sira-Ramirez and Agrawal (2004) for a review of flatness. We parameterize the motion of (6) by a flat output trajectory

$$\begin{aligned}
\gamma_1(t) &:= p_{r1}(t) = (6 + 2\cos(\omega_v t))\cos(\omega_u t)\\
\gamma_2(t) &:= p_{r2}(t) = (6 + 2\cos(\omega_v t))\sin(\omega_u t)\\
\gamma_3(t) &:= p_{r3}(t) = 2\sin(\omega_v t)\\
\gamma_4(t) &:= \psi(t) \quad = \omega_u t + \pi,
\end{aligned}$$

where the first three flat output dimensions are taken to be the position of the UAV in the global frame of reference, and the fourth output is chosen as the yaw angle in a ZYX Tait-Bryan representation, i.e., the rotation of the system about the e_3 direction in the global reference frame.

The trajectory is defined by constant angular rates $\omega_u = 0.2\pi\ (rad/s)$ and $\omega_v = 1.2\pi\ (rad/s)$, over $t \in [0, 70/6]\ (s)$, and just as in the previous examples, the initial conditions of the system are randomized according to Table 1. In Fig. 3, the resulting system response is shown in terms of the system configurations in time, plotted over the torus on whose surface the UAV is looping. In Fig. 4 the tracking error is shown over the expanded reference trajectory.

Despite the large initial errors, the system quickly converges to the reference trajectory, implying that the estimator errors converge to zero, and it is clear that $(\rho_e, \nu_e, \tilde{\rho}_e, \tilde{\nu}_e, \tilde{z}, R_e, \tilde{\omega}, \tilde{R}, \omega_e) \to (0, 0, 0, 0, 0, I, 0, I, 0)$.

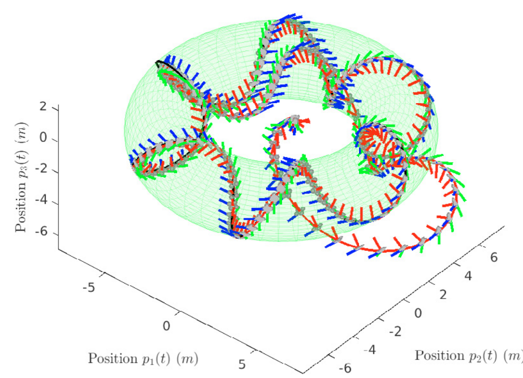

Fig. 3. Configurations while tracking the looping trajectory from a large initial control and estimate error. The initial position is close to the center of the torus.

8. CONCLUSION

In this paper, we present a novel output feedback controller for the problem of trajectory tracking with a quadrotor UAV using partial state-information, as defined in Section 3. The proposed control system comes with four main advantages when considering practical implementations. Firstly, the translational control is saturated, permitting the bounding of the virtual control signal $u(t)$ so as to comply with actuator constraints. Secondly, the controller filters the acquired measurements, thereby attenuating the effect of measurement noise in the control signals and states. Thirdly, the controller only uses information which is ubiquitous in modern UAV applications, including positional, gyroscopic, accelerometer and magnetometer measurements. It does not rely on full state information, as many UAV controllers do, but rather readily available

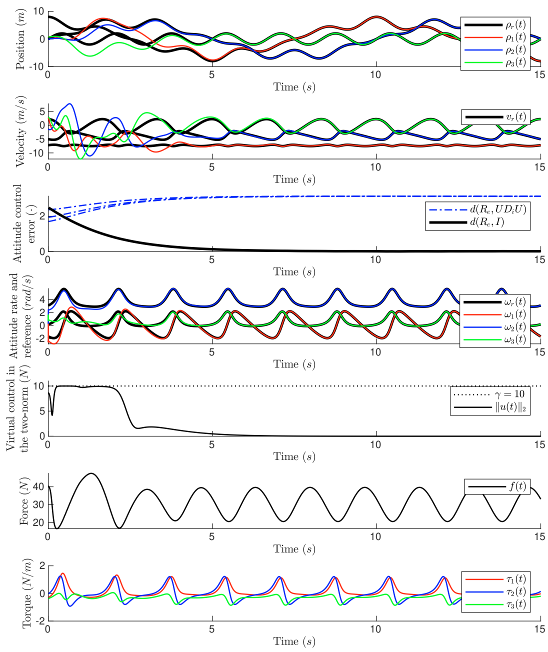

Fig. 4. From top to bottom: (i) positional reference trajectory (black) and response (red,green,blue); (ii) Velocity reference trajectory (black) and response (red,green,blue); (iii) distance to unstable (blue) and stable (black) equilibrium points in the attitude tracking error; (iv) attitude rate reference trajectory (black) and response (red,green,blue); (v) virtual control signal in the two-norm; (vi) actuating force (black); and (vii) actuating torques (red, blue, green).

measurements. Fourthly, to the best knowledge of the authors, this is the first filtered output feedback controller with proven *uniform* local exponential stability, which comes with benefits in terms of robustness to disturbances.

We conclude that the proposed controller has great practical utility, and its performance will be evaluated in a real-time implementation in our future work. Furthermore, we will investigate the possibility of removing the gyroscopic measurements to make the approach even more general.

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