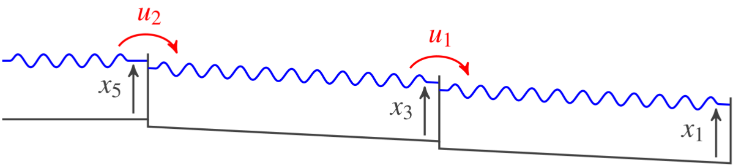
Sparse large scale control

With tools from linear algebra and an algebra of shift operators

Julia Adlercreutz

LTH, Department of automatic control

Problem with LQR



$$u(k) = -Kx(k)$$

$$K = \begin{bmatrix} 0.1000 & 0.1000 & -0.4000 & -0.4000 & 0 \\ 0.0238 & 0.0238 & 0.0238 & 0.0238 & -0.4762 \end{bmatrix}$$

Sparsity in my research

Normal LQR solution:

$$u = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix} x$$

Sparsity in my research

Normal LQR solution:

$$u = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix} x$$

My research:

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix} u = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0* & * & * & * & 0 \end{bmatrix} x$$

Shift operators

Backward shift: q^*

Shift operators

Backward shift: $q^* \left(y[0], y[1], y[2], \ldots\right) = \left(0, y[0], y[1], y[2], \ldots\right)$

Shift operators

Backward shift: $q^* (y[0], y[1], y[2], ...) = (0, y[0], y[1], y[2], ...)$

Forward shift: $q (y[0], y[1], y[2], ...) = (y[1], y[2], ...)$

Shift operators

Backward shift: $q^* y(k) = y(k-1)$

Forward shift: $qy(k) = y(k+1)$

Shift operator multiplication with scalar

$$2 \cdot q^* = 2q^*$$

Shift operator multiplication with scalar

$$2 \cdot q^* = 2q^*$$

$$2q^* \left(y[0], y[1], y[2], ...\right) = \left(0, 2y[0], 2y[1], 2y[2], ...\right)$$

Shift operator addition

$$q^* + q^* = 2q^*$$

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$$(q^* + q^*)\,(y[0], y[1], y[2], ...) = (0, 2y[0], 2y[1], 2y[2], ...)$$

Shift operator addition

$$q^* + q^* = 2q^*$$

$$(q^* + q^*)\,(y[0], y[1], y[2], \ldots) = (0, 2y[0], 2y[1], 2y[2], \ldots)$$

$$= q^*\,(y[0], y[1], y[2], \ldots) + q^*\,(y[0], y[1], y[2], \ldots)$$

Shift operator addition

$$(q^* + q)\,(y[0], y[1], y[2], \ldots) =$$
$$= q^*\,(y[0], y[1], y[2], \ldots) + q\,(y[0], y[1], y[2], \ldots)$$

Shift operator multiplication

$$q^* \cdot q^* = (q^*)^2$$

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$$q^* \left(q^* \left(y[0], y[1], y[2], ... \right) \right) = \left(0, 0, y[0], y[1], y[2], ... \right)$$

Shift operator multiplication

$$q \cdot q = q^2$$

$$q\left(q\left(y[0], y[1], y[2], ...\right)\right) = \left(y[2], ...\right)$$

Shift operator multiplication

$$q \cdot q^* \neq q^* \cdot q$$

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$$q\left(q^*\left(y[0], y[1], y[2], \ldots\right)\right) = \left(y[0], y[1], y[2], \ldots\right)$$

Shift operator multiplication

$$q \cdot q^* \neq q^* \cdot q$$

$$q\left(q^*\left(y[0], y[1], y[2], \dots\right)\right) = \left(y[0], y[1], y[2], \dots\right)$$

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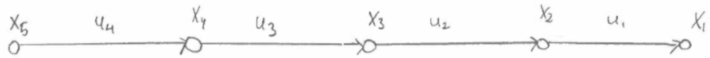
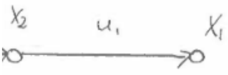
Transportation Network

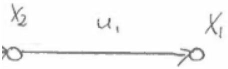


Transportation Network



$$x_1(k+1) = x_1(k) + u_1(k-1)$$

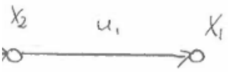
Transportation Network



$$x_1(k+1) = x_1(k) + u_1(k-1)$$

$$x_1(k) = x_1(k-1) + u_1(k-2)$$

Transportation Network

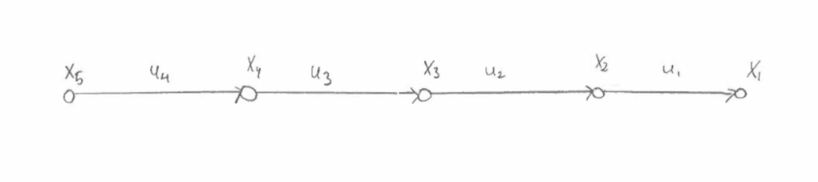


$$x_1(k+1) = x_1(k) + u_1(k-1)$$

$$x_1(k) = x_1(k-1) + u_1(k-2)$$

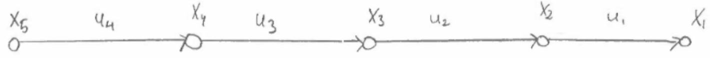
$$x_1(k) = q^* x_1(k) + (q^*)^2 u_1(k)$$

Transportation Network



$$x = q^* x + q^* \begin{bmatrix} q^* & 0 & 0 & 0 \\ -1 & q^* & 0 & 0 \\ 0 & -1 & q^* & 0 \\ 0 & 0 & -1 & q^* \\ 0 & 0 & 0 & -1 \end{bmatrix} u + [x_{init}, 0, 0, 0, \ldots]$$

Transportation Network



$$(1-q^*)x = q^* \begin{bmatrix} q^* & 0 & 0 & 0 \\ -1 & q^* & 0 & 0 \\ 0 & -1 & q^* & 0 \\ 0 & 0 & -1 & q^* \\ 0 & 0 & 0 & -1 \end{bmatrix} u + [x_{init}, 0, 0, 0, ...]$$

Linear Quadratic Regulator Problem

LQR Problem

$$\min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

Subject to System dynamics

Solution

$$u(k) = -Kx(k)$$

LQR as Least squares

$$\min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} x(k)^T x(k)$$

$$\text{Subject to} \quad (1 - q^*)x = Mu + d$$

LQR as Least squares

$$\min_{u(0),u(1),\ldots} \|x\|^2_{l2+}$$

$$\text{Subject to} \quad \left[(1-q^*)I \quad -M\right] \begin{bmatrix} x \\ u \end{bmatrix} = d$$

LQR as Least squares

$$\min_{u(0),u(1),\ldots} \|x\|_{l2+}^2$$

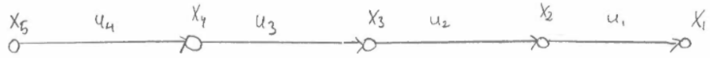
$$\text{Subject to} \quad [(1-q^*)I \quad -M] \begin{bmatrix} x \\ u \end{bmatrix} = d$$

Closed form solution

$$M^* M \hat{v} = -M^* w_{init}$$

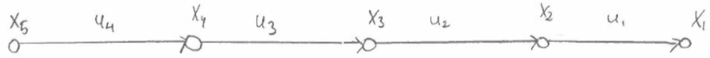
$\hat{v} \sim$ Optimal control law
$w_{init} \sim$ Initial conditions

LDL factorisation



$$M^*M\hat{v} = -M^*w_{init}$$

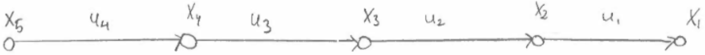
LDL factorisation



$$M^* M \hat{v} = -M^* w_{init}$$

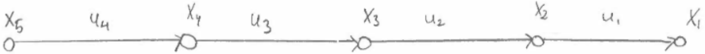
$$LDL^* \hat{v} = -M^* w_{init}$$

LDL factorisation



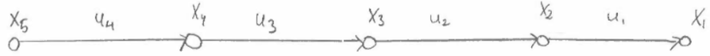
$$N = M^*M = \begin{pmatrix} 2 & -q^* & 0 & 0 \\ -q & 2 & -q^* & 0 \\ 0 & -q & 2 & -q^* \\ 0 & 0 & -q & 2 \end{pmatrix}$$

LDL factorisation



$$N = M^*M = \left(\begin{array}{c|ccc} 2 & -q^* & 0 & 0 \\ \hline -q & 2 & -q^* & 0 \\ 0 & -q & 2 & -q^* \\ 0 & 0 & -q & 2 \end{array}\right)$$

LDL factorisation



$$N = M^* M = \left(\begin{array}{c|ccc} 2 & -q^* & 0 & 0 \\ \hline -q & 2 & -q^* & 0 \\ 0 & -q & 2 & -q^* \\ 0 & 0 & -q & 2 \end{array}\right)$$

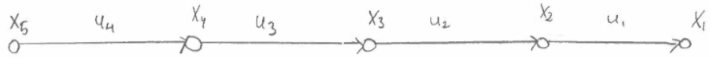
$$N_{red} = N_{22} - N_{21} N_{11}^{-1} N_{12}$$

LDL

$$N = \begin{bmatrix} 1 & 0 \\ N_{21}N_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} N_{11} & 0 \\ 0 & N_{red} \end{bmatrix} \begin{bmatrix} 1 & N_{11}^{-1}N_{12} \\ 0 & I \end{bmatrix}$$

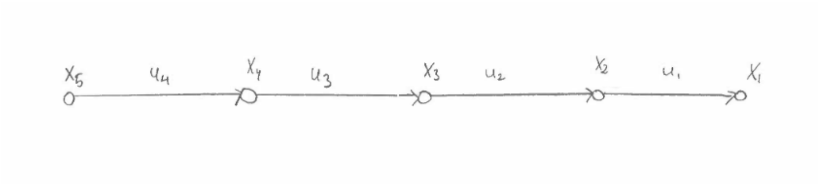
$$N_{red} = N_{22} - N_{21}N_{11}^{-1}N_{12}$$

LDL factorisation



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2}q & 1 & 0 & 0 \\ 0 & -\frac{2}{3}q & 1 & 0 \\ 0 & 0 & -\frac{3}{4}q & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2}q & 1 & 0 & 0 \\ 0 & -\frac{2}{3}q & 1 & 0 \\ 0 & 0 & -\frac{3}{4}q & 1 \end{bmatrix}^* \hat{v} = -M^* w_{init}$$

LDL factorisation



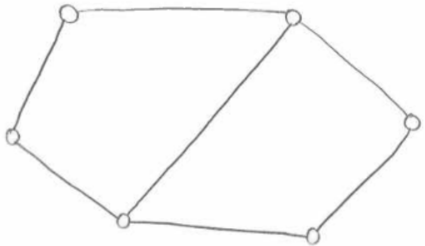
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2}q & 1 & 0 & 0 \\ 0 & -\frac{2}{3}q & 1 & 0 \\ 0 & 0 & -\frac{3}{4}q & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2}q & 1 & 0 & 0 \\ 0 & -\frac{2}{3}q & 1 & 0 \\ 0 & 0 & -\frac{3}{4}q & 1 \end{bmatrix}^* \hat{v} = -M^* w_{init}$$

My research:

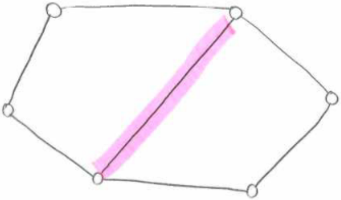
$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix} u = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0* & * & * & * & 0 \end{bmatrix} x$$

GRAPH THEORY

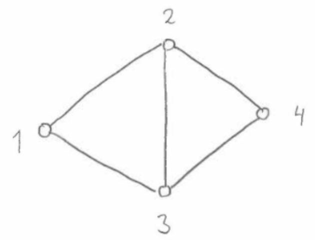
Chords



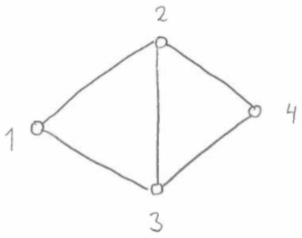
Chords



Adjacency matrix

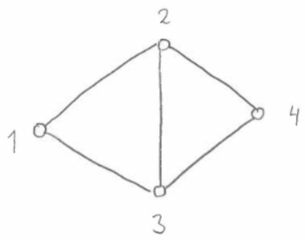


Adjacency matrix



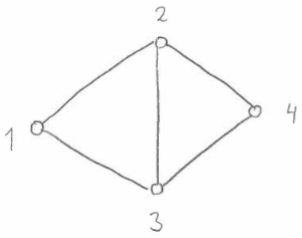
$$A = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Adjacency matrix



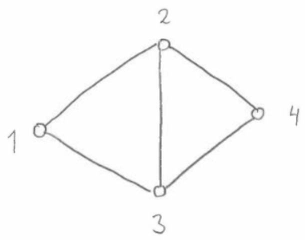
$$A = \begin{bmatrix} * & * & * & 0 \\ & & & \\ & & & \end{bmatrix}$$

Adjacency matrix



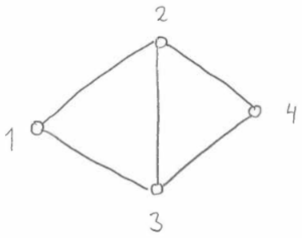
$$A = \begin{bmatrix} * & * & * & 0 \\ * & * & * & * \\ & & & \\ & & & \end{bmatrix}$$

Adjacency matrix



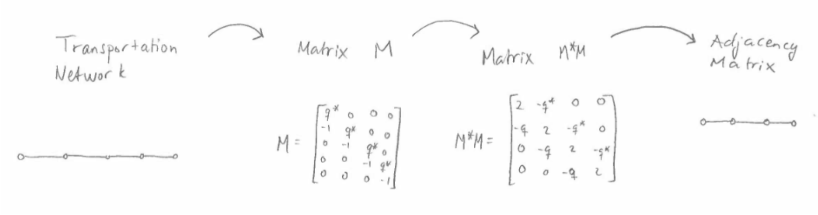
$$A = \begin{bmatrix} * & * & * & 0 \\ * & * & * & * \\ * & * & * & * \\ & & & \end{bmatrix}$$

Adjacency matrix

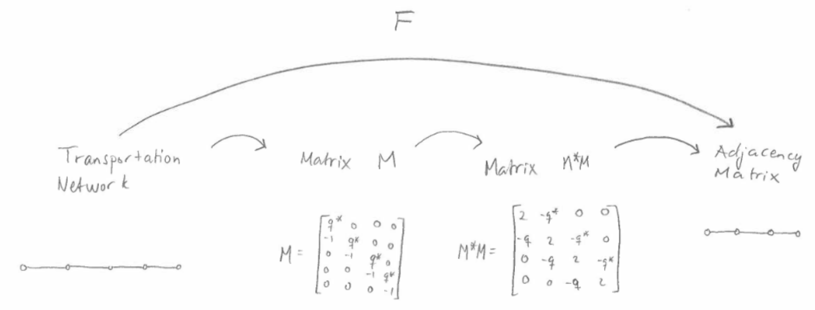


$$A = \begin{bmatrix} * & * & * & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

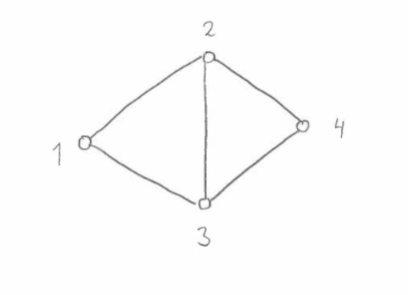
Mapping


Transportation Network
Matrix M
Matrix M*M
Adjacency Matrix

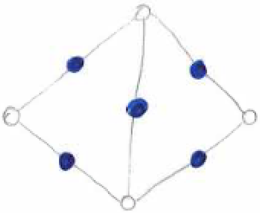
Mapping


F
Transportation Network
Matrix M
Matrix M*M
Adjacency Matrix
M =
M*M =

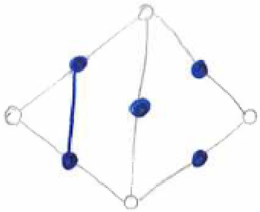
Mapped graph



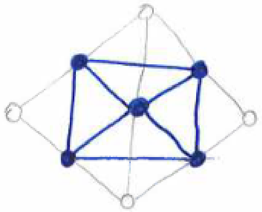
Mapped graph



Mapped graph



Mapped graph



Theorem

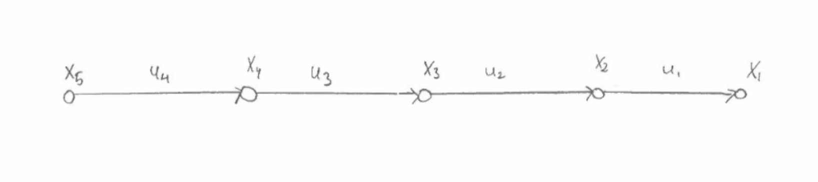
Theorem 1

Let G be an undirected graph. The following are equivalent

(i) G has no cycles longer than three

(ii) $F(G)$ is chordal.

LDL factorisation PROBLEM



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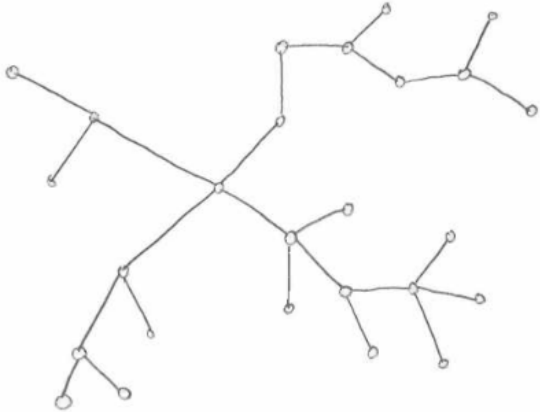
$$N_{red} = N_{22} - N_{21}N_{11}^{-1}N_{12}$$

Cool graph



Even cooler graph