On Scalable Methods for H_∞ control

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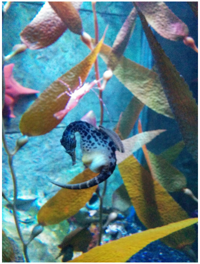
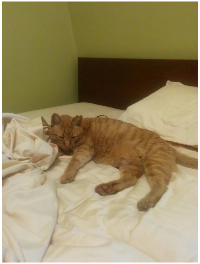
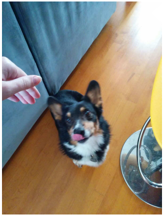
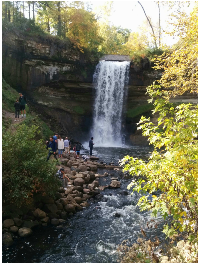
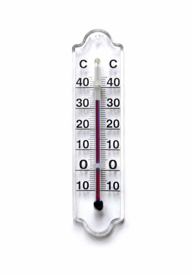








The need for scalable control



Outline

An interesting observation

$\rightarrow$ Simple and scalable form of an optimal static H_∞-controller

Ongoing work

An interesting observation

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} u + w$$

- Performance output (x, u)
- Design of static state feedback control, $u = Lx$

Synthesis of static H_∞ controllers

Problem formulation: find a matrix L that minimizes $\|G_L\|_\infty$ where G_L is the closed-loop transfer function.

- State feedback: exist static controllers that are optimal
- Not unique

Synthesis tools: ARE, LMI.

Example cont'd

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} u + w$$

Examples of optimal controllers L:

$$L_1 = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & -1/2 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0.93 & -0.11 & 0.00 \\ -0.05 & -0.17 & -0.01 \\ 0.04 & 0.16 & -0.26 \end{pmatrix}$$

Example cont'd

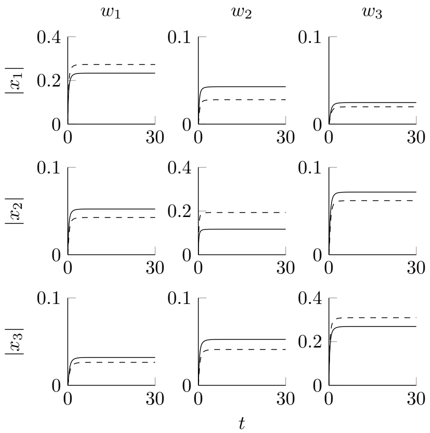
$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{B} u + w$$

Examples of minimizing controllers L:

$$L_1 = \underbrace{\begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & -1/2 \end{pmatrix}}_{=B^T A^{-1}} \quad L_2 = \begin{pmatrix} 0.93 & -0.11 & 0.00 \\ -0.05 & -0.17 & -0.01 \\ 0.04 & 0.16 & -0.26 \end{pmatrix}$$

Performance

L_1 solid line, L_2 dashed line.



LMI approach

Given

$$G_L(s) = \begin{pmatrix} I \\ L \end{pmatrix} (sI - (A + BL))^{-1}$$

the following conditions are equivalent by the K-Y-P lemma

i The matrix $A + BL$ is Hurwitz and $\|G_L\|_\infty < \gamma$

ii There exists a matrix $P \succ 0$ such that

$$\begin{pmatrix} I + L^T L & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (A + BL)^T P + P(A + BL) & P \\ P & -\gamma^2 I \end{pmatrix} \prec 0$$

Optimization problem

Variables P and L.

$$\begin{aligned}
&\text{minimize } \gamma \\
&\text{subject to } P \succ 0 \\
&\begin{pmatrix} I + L^T L & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (A+BL)^T P + P(A+BL) & P \\ P & -\gamma^2 I \end{pmatrix} \prec 0
\end{aligned}$$

Non-convex in L and P.

Optimization problem

- multiply LMI with $\mathrm{diag}\left(P^{-1}, I\right)$ from left and right
- Perform variable transformation $X = P^{-1}$ and $Y = LP^{-1}$.

Variables X and Y.

$$\begin{aligned}&\text{minimize } \gamma\\&\text{subject to } X \succ 0\\&\begin{pmatrix} X^2 + Y^T Y & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} XA^T + Y^T B^T + AX + BY & I \\ I & -\gamma^2 I \end{pmatrix} \prec 0\end{aligned}$$

Convex in X and Y.

Optimal X and Y

Rewrite LMI by Schur's complement lemma

$$X^2 + XA^T + AX + Y^TY + Y^TB^T + BY + \gamma^{-2}I \prec 0$$

which is equivalent to

$$(X+A)(X+A)^T - AA^T + (Y^T+B)(Y^T+B)^T - BB^T + \gamma^{-2}I \prec 0$$

Assumptions and one choice of Y

Assumptions

- (A, B) is stabilizable
- A is Hurwitz

Choose $Y = -B^T$,

$$(X + A)(X + A)^T - AA^T - BB^T + \gamma^{-2} I \prec 0$$

Lower bound on γ

For any A and B

$$\|G_L\|_\infty \geq \frac{1}{\lambda_{min}(AA^T + BB^T)}.$$

When is equality achieved?

Symmetric A-matrix

- A is symmetric and Hurwitz

$$(X + A)(X + A)^T - AA^T - BB^T + \gamma^{-2} I \prec 0$$

Choose $X = -A \rightarrow L = YX^{-1} = B^T A^{-1}$.

Back to example...

$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{B} u + w$$

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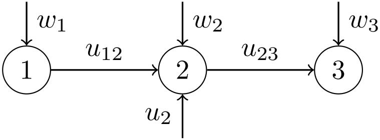
$$L_1 = \underbrace{\begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & -1/2 \end{pmatrix}}_{=B^T A^{-1}} \quad L_2 = \begin{pmatrix} 0.93 & -0.11 & 0.00 \\ -0.05 & -0.17 & -0.01 \\ 0.04 & 0.16 & -0.26 \end{pmatrix}$$

Distributed controller given sparse plant



$$L_1 = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & -1/2 \end{pmatrix} \quad \rightarrow \quad u = \begin{pmatrix} u_{12} \\ u_2 \\ u_{23} \end{pmatrix} = \begin{pmatrix} x_1 - x_2/3 \\ -x_2/3 \\ x_2/3 - x_3/2 \end{pmatrix}$$

$u_{12} = x_1 - x_2/3 < 0$, flow from 2 to 1 when $x_2 > 3x_1$.

Comparison with other methods

- Regular ARE and LMI optimization does not give $L = B^T A^{-1}$
- Sparsity constraints on L, might not get optimal controller
- Scalability

Coordination in the H_∞ framework

System with ν subsystems

$$\dot{x}_i = A_i x_i + B_i u_i + w_i, \quad \nu = 1, \dots, \nu$$

with coordination constraint $u_1 + u_2 + \cdots + u_\nu = 0$.

Optimal control law:

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1}$$

Comparison with H_2-method [Madjidian and Mirkin, 2014]

- Heterogeneous subsystems
- Necessary that A_i are symmetric and Hurwitz

Ongoing work

- Diagonal X in $L = B^T X^{-1}$ when A is Metzler
- Output feedback
- Non-linear state feedback
- Other norm