Model reduction of switched affine systems:
a method based on balanced truncation and randomized optimization

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Problem statement

Problem: Approximating a hybrid system by means of a simpler model

Goal: Obtain a simpler model able to mimic the system behavior over some finite horizon $\mathcal{T}$ aimed at system verification

Outline

1. **Model Order Reduction for linear systems**
2. **Modeling Framework**
3. **System Reduction**
4. **A Randomized Method for Model Order Selection**
5. **Application Example**

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1. **Model Order Reduction for linear systems**
2. Modeling Framework
3. System Reduction
4. A Randomized Method for Model Order Selection
5. Application Example

Basic terminology

Let $\mathcal{S}$ be a continuous-time linear time-invariant dynamic system described in state-space

$$\mathcal{S} : \begin{cases} \dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{B}u(t) \\ y(t) = \mathcal{C}\xi(t) + \mathcal{D}u(t) \end{cases} \qquad \Leftrightarrow \quad \mathcal{S} : \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}.$$

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Definition 1 (Gramians).

The *controllability* and *observability Gramians* of $\mathcal{S}$ are defined as

$$\mathcal{W}_c(t) = \int_0^t e^{\mathcal{A}\tau}\mathcal{B}\mathcal{B}' e^{\mathcal{A}'\tau} \, d\tau$$

$$\mathcal{W}_o(t) = \int_0^t e^{\mathcal{A}'\tau}\mathcal{C}'\mathcal{C} \, e^{\mathcal{A}\tau} \, d\tau.$$

Balanced realization

Definition 2 (Balanced system).

System $\mathcal{S}$ is *balanced* if $\mathcal{W}_c = \mathcal{W}_o$.
Furthermore, $\mathcal{S}$ is *principal-axis balanced* if $\mathcal{W}_c = \mathcal{W}_o = \Sigma$, with

$$\Sigma = \operatorname{diag}\left\{\sigma_1, \sigma_2, \ldots, \sigma_n\right\},$$

where σ_i are the Hankel singular values of $\mathcal{S}$, listed in decreasing order.

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where σ_i are the Hankel singular values of $\mathcal{S}$, listed in decreasing order.

A balanced realization is obtained by determining a balancing transformation matrix T such that

$$\begin{cases} W_c = T\mathcal{W}_c T^* \\ W_o = T^{-*}\mathcal{W}_o T^{-1} \end{cases} \quad \Rightarrow \; W_c\, W_o = T\left(\mathcal{W}_c\mathcal{W}_o\right) T^{-1} = \Sigma^2,$$

Balanced realization

The state vector can be transformed in its balanced form according to

$$T\xi(t) = x(t)$$

while the state vector of the balanced realization can be partitioned as $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 \in \mathbb{R}^{n_r}$ and $x_2 \in \mathbb{R}^{n-n_r}$.

A balanced realization is thus obtained as

$$S : \begin{pmatrix} T\mathcal{A}T^{-1} & T\mathcal{B} \\ \mathcal{C}T^{-1} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{pmatrix}.$$

Balanced truncation

A reduced order model can be obtained by setting $\dot{x}_2 = 0$, yielding

$$S_r : \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{pmatrix}.$$

An estimate of the neglected state x_2 is given by

$$\widehat{x}_2 = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u,$$

which corresponds to the condition $\dot{x}_2 = 0$

Order selection

To select the order n_r of the reduced system, one can choose $\gamma \in [0, 1]$ and set

$$n_r = \underset{i \in \{1,2,\ldots,n\}}{\arg\min} \left(\psi(i) \leq \gamma\right),$$

where $\psi : \{1, 2, \ldots n\} \to [0, 1)$ is defined based on the Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ of system S as follows:

$$\psi(i) = 1 - \frac{\sum_{j=1}^{i} \sigma_j}{\sum_{j=1}^{n} \sigma_j}.$$

Outline

Switched Affine systems

Switched Affine (SA) systems

- discrete state component $q_a \in Q = \{1, 2, \ldots, m\}$
- continuous component $\xi_a \in \Xi_a = \mathbb{R}^n$ and output $y_a \in Y_a = \mathbb{R}^p$, evolving according to

$$\begin{cases} \dot{\xi}_a(t) = \mathcal{A}_{q_a}\xi_a(t) + \mathcal{B}_{q_a}u(t) + f_{q_a} \\ y_a(t) = \mathcal{C}_{q_a}\xi_a(t) + g_{q_a}. \end{cases}$$

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- A collection of polyhedra

$$\bigcup_{i \in Q} \{Dom_{a,i} \subseteq Y_a \times U,\ i \in Q\} = Y_a \times U$$

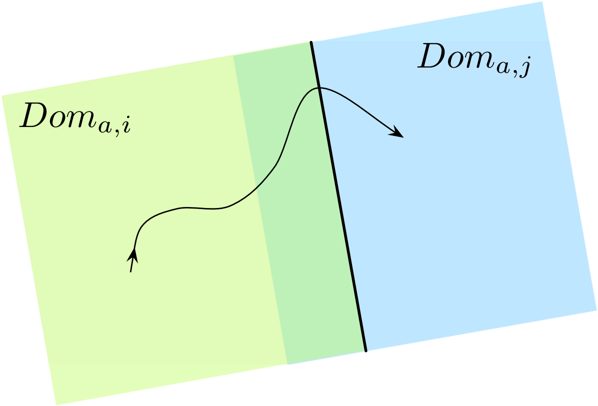
- Each polyhedron $Dom_{a,i}$ is defined through a system of r_i linear inequalities

$$Dom_{a,i} = \{(y_a, u) \in Y_a \times U : G_i^{y_a} y_a + G_i^u u \leq G_i\},$$

with $G_i^{y_a} \in \mathbb{R}^{r_i \times p}$, $G_i^u \in \mathbb{R}^{r_i \times m}$ and $G_i \in \mathbb{R}^{r_i}$.

Switched Affine systems

- A transition from mode $i \in Q$ to mode $j \in Q$ occurs as soon as
  - $(y_a, u) \notin Dom_{a,i}$, and
  - $(y_a, u) \in Dom_{a,j}$
- Identity reset maps



Outline

System Reduction

Procedure

1. Reformulation of the Switched Affine (SA) system as a Switched Linear (SL) one
2. Reduction of the SL system
   1. **Redefinition of reset maps**
   2. **Reduction and order selection of the continuous component**
3. Reconstruction of the SA system output

Reformulation of the SA system as a SL one

Let $\xi \in \Xi = \Xi_a$, and $y \in Y = Y_a$ evolve as

$$\begin{cases} \dot{\xi}(t) = \mathcal{A}_q \xi(t) + \mathcal{B}_q u(t) \\ y(t) = \mathcal{C}_q \xi(t) \end{cases}$$

and let

$$\bar{\xi}_{a,q} = -\mathcal{A}_q^{-1} f_q$$
$$\bar{y}_{a,q} = \mathcal{C}_q \bar{\xi}_{a,q} + g_q$$

A transition from mode $i \in Q$ to mode $j \in Q$ occurs as soon as

- $(y + \bar{y}_{a,i}, u) \notin Dom_i$, and
- $(y + \bar{y}_{a,i}, u) \in Dom_j$

where $Dom_q = Dom_{a,q}$, $q \in Q$.
When a transition occurs at time t^-, then, ξ is reset as follows

$$\xi(t) = \xi(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j}.$$

Reformulation of the SA system as a SL one

Proposition.

Suppose that SA and SL have

- $\xi_a(0) = \xi_{a,0}$ *and* $\xi(0) = \xi_{a,0} - \bar{\xi}_{a,q_{a,0}}$,
- $q_a(0) = q_{a,0}$ *and* $q(0) = q_{a,0}$, *and*
- *both fed by the same input* $u(t)$, $t \in [0, \mathcal{T}]$

Then, the execution of ξ_a, q_a *and* y_a *over* $[0, \mathcal{T}]$ *can be recovered as*

$$q_a(t) = q(t)$$
$$\xi_a(t) = \xi(t) + \bar{\xi}_{a,q(t)}$$
$$y_a(t) = y(t) + \bar{y}_{a,q(t)}.$$

Reduction of the SL system

We associate to each mode $q_r \in Q$ a reduced model of order $n_{r,q} < n$

$$\begin{cases} \dot{x}_r(t) = A_{r,q_r} x_r(t) + B_{r,q_r} u(t) \\ \hat{y}(t) = C_{r,q_r} x_r(t) + D_{r,q_r} u(t) \end{cases}$$

A transitions from mode i to mode j when

- $(\hat{y} + \bar{y}_{a,i}, u) \notin Dom_i$, and
- $(\hat{y} + \bar{y}_{a,i}, u) \in Dom_j$

When a transition occurs at time t^-, then, x_r is reset as follows

$$x_r(t) = L_{ji} x_r(t^-) + M_{ji} u(t^-) + N_{ji}.$$

Estimate of the state variables

1. The estimate $\hat{x}$ is reconstructed from the reduced state x_r as

$$\hat{x} = \begin{bmatrix} x_r \\ -A_{i,22}^{-1} A_{i,21} x_r - A_{i,22}^{-1} B_{i,2} u \end{bmatrix} = \begin{bmatrix} I_{n_{r,i} \times n_{r,i}} \\ -A_{i,22}^{-1} A_{i,21} \end{bmatrix} x_r + \begin{bmatrix} \mathbf{0}_{n_{r,i} \times 1} \\ -A_{i,22}^{-1} B_{i,2} \end{bmatrix} u$$

which can be rewritten in compact form as

$$\hat{x} = H_i \, x_r + K_i \, u,$$

2. The estimate $\hat{\xi}$ of the state of the SL system associated with mode $i \in Q$:

$$\hat{\xi} = T_i^{-1} \hat{x},$$

obtained from $\hat{x}$ through the balanced transformation matrix T_i.

Reset map (a) from Mazzi et al., CDC 2008

The reset map of the reduced system is

$$x_r(t) = E_{n_{r,j}} \hat{x}(t) = E_{n_{r,j}} T_j \hat{\xi}(t)$$

where $E_{n_{r,j}}$ is a matrix extracting the first $n_{r,j}$ rows from $\hat{x}(t)$, being $n_{r,j}$ the dimension of x_r in mode j.

$$\begin{aligned} \hat{\xi}(t) &= \hat{\xi}(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} = T_i^{-1} \hat{x}(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \\ &= T_i^{-1} H_i \, x_r(t^-) + T_i^{-1} K_i \, u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j}, \end{aligned}$$

yielding

$$x_r(t) = E_{n_{r,j}} T_j \left(T_i^{-1} H_i x_r(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right)$$

Reset map (b) best reproducing the output free response

We choose the reset map as

$$x_r(t) = \Psi_j^{(\tau)} \hat{\xi}(t)$$

and $\Psi_j^{(\tau)}$ so as to minimize over an horizon of length $\tau > 0$ (possibly $\tau = \infty$)

$$J = \int_0^{\tau} \|y_{fr,j}(t) - \hat{y}_{fr,j}(t)\|^2 \, \mathrm{d}t,$$

Reset map (b) best reproducing the output free response

We choose the reset map as

$$x_r(t) = \Psi_j^{(\tau)} \hat{\xi}(t)$$

Proposition.

Matrix $\Psi_j^{(\tau)}$ *minimizing* J *for any* $\hat{\xi}$ *is given by* $\Psi_j^{(\tau)} = \mathcal{W}_{r,o,j}^{-1}(\tau)\mathcal{W}_{\times,j}(\tau)$, *where*

$$\mathcal{W}_{r,o,j}(\tau) = \int_0^\tau (e^{A_{r,j}t})' C_{r,j}' C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t$$

$$\mathcal{W}_{\times,j}(\tau) = \int_0^\tau (e^{\mathcal{A}_j t})' \mathcal{C}_j' C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t,$$

Reset map (b) best reproducing the output free response

We choose the reset map as

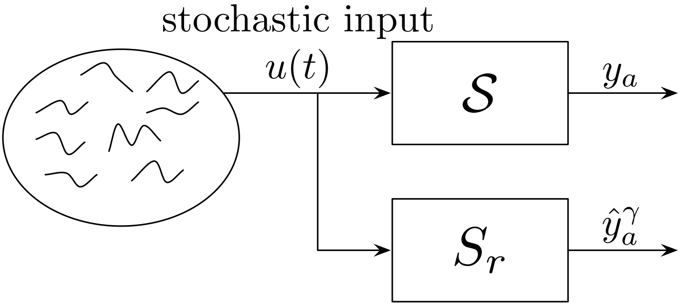
$$x_r(t) = \Psi_j^{(\tau)} \hat{\xi}(t)$$

yielding

$$\begin{aligned} x_r(t) =& \Psi_j^{(\tau)} \hat{\xi}(t) = \Psi_j^{(\tau)} \left(\hat{\xi}(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) = \\ =& \Psi_j^{(\tau)} \left(T_i^{-1} \hat{x}(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \\ =& \Psi_j^{(\tau)} \left(T_i^{-1} H_i x_r(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \\ =& \mathcal{W}_{r,o,j}^{-1}(\tau) \mathcal{W}_{\times,j}(\tau) \left(T_i^{-1} H_i x_r(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \end{aligned}$$

Outline

Problem description



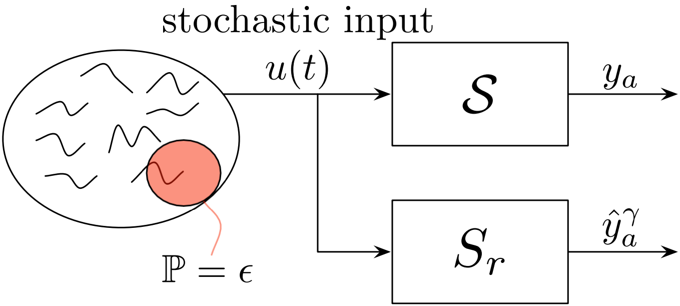
- Model S_r is fed with the same stochastic input $u(t)$ as the original system $\mathcal{S}$ and tries to reproduce y_a through $\hat{y}_a^\gamma$ along a time horizon $\mathcal{T}$

Quality of the approximation

Model S_r is a ρ-approximation up to level $1-\epsilon$ of system $\mathcal{S}$ if

$$\mathbb{P}\{d_{\mathcal{T}}(y_a, \hat{y}_a^\gamma) \leq \rho\} \geq 1-\epsilon$$

Notion adopted in [Julius and Pappas, IEEE TAC 2009] and [Abate and Prandini, IEEE CDC 2011]



A Randomized Method for Model Order Selection

For each $\gamma \in \Gamma \subset [0,1]$, the approximation quality $\rho_\gamma^\star$ of the reduced order model with parameter γ is the solution to the following chance-constrained optimization problem:

$$CCP_\gamma : \min_\rho \rho$$
$$\text{subject to: } \mathbb{P}\{d_{\mathcal{T}}(y_a, \hat{y}_a^\gamma) \leq \rho\} \geq 1 - \epsilon.$$

A Randomized Method for Model Order Selection

For each $\gamma \in \Gamma \subset [0, 1]$, the approximation quality $\rho^\star_\gamma$ of the reduced order model with parameter γ is the solution to the following chance-constrained optimization problem:

$$CCP_\gamma : \min_\rho \rho$$
$$\text{subject to: } \mathbb{P}\{d_{\mathcal{T}}(y_a, \hat{y}^\gamma_a) \leq \rho\} \geq 1 - \epsilon.$$

Remark.

As argued in [Abate & Prandini, IEEE CDC 2011], the directional Hausdorff distance

$$d_{\mathcal{T}}(y_a, \hat{y}^\gamma_a) = \sup_{t \in \mathcal{T}} \inf_{\tau \in \mathcal{T}} \|y_a(t) - \hat{y}^\gamma_a(\tau)\|$$

is a sensible choice for $d_{\mathcal{T}}(y_a, \hat{y}^\gamma_a)$ *when performing probabilistic verification.*

Randomized solution [Campi & Garatti, JOTA 2011]

Extract N realizations of the stochastic input $u^{(i)}(t)$, $t \in \mathcal{T}$, and consider only the corresponding constraints

$$
\begin{aligned}
& CCP_\gamma : \min_\rho \rho \\
& \text{subject to: } d_{\mathcal{T}}(y_a^{(1)}, \hat{y}_a^{\gamma,(1)}) := \hat{\rho}^{(1)} \leq \rho \\
& \qquad d_{\mathcal{T}}(y_a^{(2)}, \hat{y}_a^{\gamma,(2)}) := \hat{\rho}^{(2)} \leq \rho \\
& \qquad \dots \\
& \qquad d_{\mathcal{T}}(y_a^{(i)}, \hat{y}_a^{\gamma,(i)}) := \hat{\rho}^{(i)} \leq \rho \\
& \qquad \dots \\
& \qquad d_{\mathcal{T}}(y_a^{(N)}, \hat{y}_a^{\gamma,(N)}) := \hat{\rho}^{(N)} \leq \rho
\end{aligned}
$$

Randomized solution [Campi & Garatti, JOTA 2011]

Determine the $\lfloor \eta N \rfloor$ largest values of $\{\hat{\rho}^{(i)},\ i = 1, 2, \ldots, N\}$ and remove them so as to improve the cost

$$CCP_\gamma : \min_\rho \rho$$

subject to:

$$d_\mathcal{T}(y_a^{(1)}, \hat{y}_a^{\gamma,(1)}) := \hat{\rho}^{(1)} \leq \rho$$

$$d_\mathcal{T}(y_a^{(2)}, \hat{y}_a^{\gamma,(2)}) := \hat{\rho}^{(2)} \leq \rho$$

$$\ldots$$

$$d_\mathcal{T}(y_a^{(i)}, \hat{y}_a^{\gamma,(i)}) := \hat{\rho}^{(i)} \leq \rho$$

$$\ldots$$

$$d_\mathcal{T}(y_a^{(N)}, \hat{y}_a^{\gamma,(N)}) := \hat{\rho}^{(N)} \leq \rho$$

Randomized solution: Scenario Theory

Proposition.

Select a confidence parameter $\beta \in (0,1)$ and an empirical violation parameter $\eta \in (0,\epsilon)$. If N is such that

$$\sum_{i=0}^{\lfloor \eta N \rfloor} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \frac{\beta}{|\Gamma|},$$

then, the solution $\hat{\rho}^\star_\gamma$, $\gamma \in \Gamma$, of the presented algorithm satisfies

$$\mathbb{P}\{d_{\mathcal{T}}(y_a, \hat{y}^\gamma_a) \leq \hat{\rho}^\star_\gamma\} \geq 1-\epsilon, \ \forall \gamma \in \Gamma,$$

with probability at least $1-\beta$.

Role of the parameter β

As $\beta \to 0$:

- $\hat{\rho}_\gamma^\star$ is certainly feasible for the original CCP
- $N \to \infty$

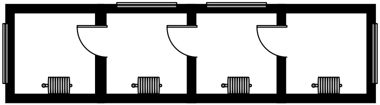
N increases with the logarithm of β [Alamo et al., 2010]

$$N \geq \frac{1}{\epsilon}\left(1 + \log\left(\frac{1}{\beta}\right) + \sqrt{2\log\left(\frac{1}{\beta}\right)}\right)$$

By choosing $\beta = 10^{-10}$, feasibility is guaranteed with a confidence that is in practice 1.

Outline

Application Example – the considered system

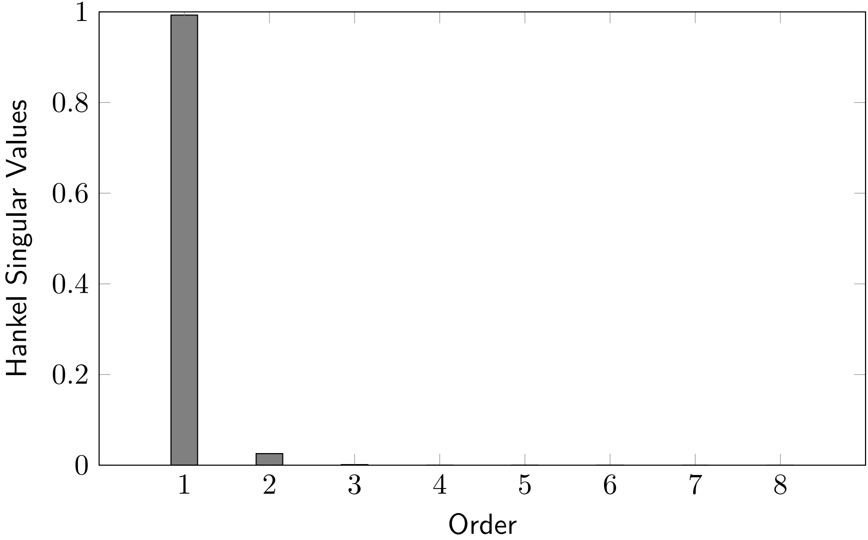


$$\begin{cases} \phi_i \dot{T}_i = \sum_{j \neq i} S_{r,ij} k_{ij} \left(T_j - T_i \right) + S_{e_i} k_{e,i} \left(T_{\text{ext}} - T_i \right) + \kappa_i \theta_i \\ \tau_{h,i} \dot{\theta}_i = -\theta_i + h_i \cdot p_i - \chi_i T_{\text{ext}} \end{cases}$$

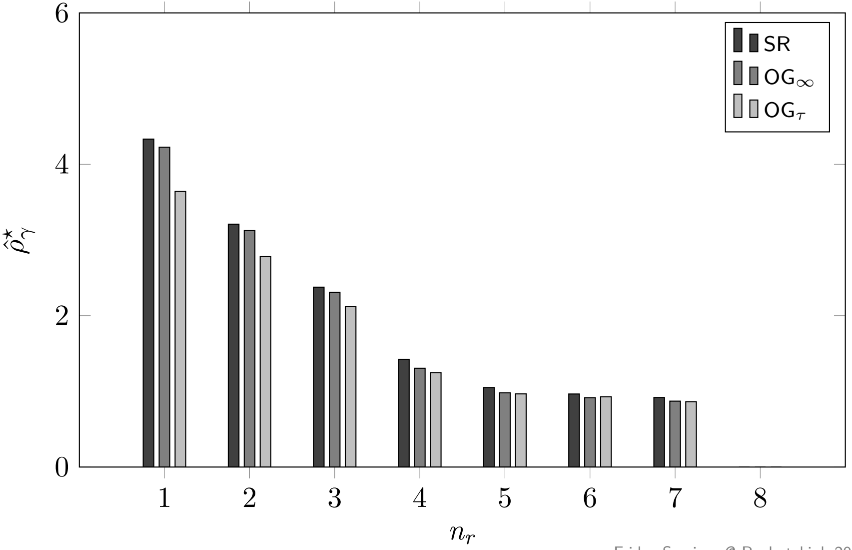
with a switching control policy

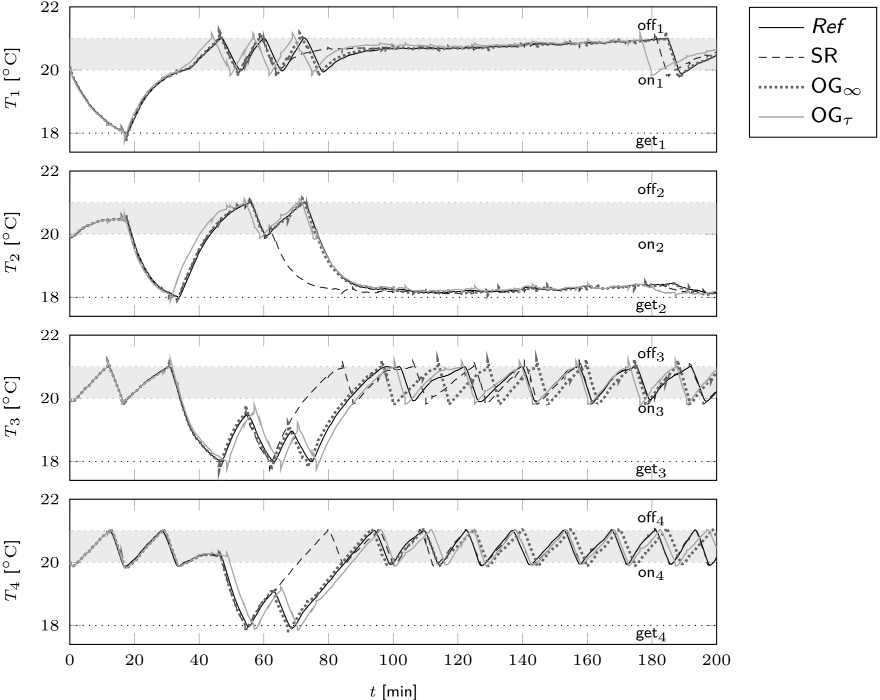
- **Room policy:** the heater in room i is on if $T_i \leq on_i$ and off if $T_i \geq off_i$,
- **Building policy:** a heater is moved from room j to an adjacent room i if the following holds
  - room i has no active heater;
  - room j has an active heater;
  - temperature $T_i \leq \text{get}_i$;
  - the difference $T_j - T_i \geq \text{dif}_i$.

Order selection – Hankel singular values



Order selection – Randomized approach





T_1 [°C]
T_2 [°C]
T_3 [°C]
T_4 [°C]
22
20
18
off$_1$
on$_1$
get$_1$
off$_2$
on$_2$
get$_2$
off$_3$
on$_3$
get$_3$
off$_4$
on$_4$
get$_4$
0
20
40
60
80
100
120
140
160
180
200
t [min]
Ref
SR
OG$_\infty$
OG$_\tau$

Conclusion and Future work

Contribution

- **State reset maps** that make the reduced model best reproduce the **free response** of the original system,
- **Randomized procedure** for model **order selection**.

- Developed extension to switching systems with **dwell time** τ_D and the approximated dynamics has a settling time smaller than τ_D.
- Future work: Study exogenous switching signal, possibly probabilistic, e.g., Markov jump linear systems.

Thank you for the attention

Questions, suggestions, comments are welcome!