Symbolic Elimination Techniques for Dynamic Optimization

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Outline

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1. Overview
2. Tearing
3. Results

Me

- Started February 2012
- Working on JModelica.org: Open-source framework for large-scale dynamic optimization
- Looking for ways to make dynamic optimization algorithms more
  - efficient
  - reliable
  - accessible

Dynamic optimization

- Optimization problems with differential equations as constraints
- Applications include
  - optimal control (open or closed loop)
  - parameter estimation or optimization
  - state estimation (moving horizon estimation)
  - experiment design

JModelica.org

Optimal control

$$
\begin{array}{ll}
\text{minimize} & \phi(t_f, x(t_f)) + \int_0^{t_f} L(x(t), u(t))\,\mathrm{d}t, \\
\text{with respect to} & t_f, x, u, \\
\text{subject to} & \dot{x} = f(x(t), u(t)), \\
& x(0) = x_0, \\
& g_i(t, x(t), u(t)) \leq 0, \\
& \psi(x(t_f)) = 0, \\
& \forall t \in [0, t_f].
\end{array}
$$

Optimal control with DAE

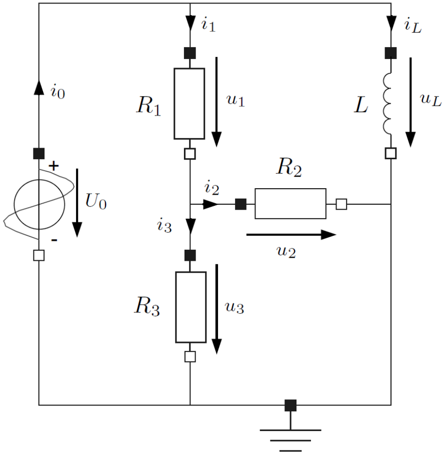
Differential-algebraic equation (DAE) instead of explicit ODE:

$$
\begin{aligned}
\text{minimize} \quad & \phi(t_f, x(t_f), y(t_f)) + \int_0^{t_f} L(x(t), y(t), u(t))\,\mathrm{d}t, \\
\text{with respect to} \quad & t_f, x, y, u, \\
\text{subject to} \quad & F(\dot{x}(t), x(t), y(t), u(t)) = 0, \\
& x(0) = x_0, \\
& g_i(t, x(t), y(t), u(t)) \leq 0, \\
& \psi(x(t_f), y(t_f)) = 0, \\
& \forall t \in [0, t_f].
\end{aligned}
$$

Simulation of DAEs

- DAE systems can be simulated with specialized DAE solvers
- Common to instead transform (reduce) the DAE to an ODE and apply ODE solvers
- These transformations have many benefits, but a few drawbacks

Example



DAE

$$U_0 = \sin(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_3 = R_3 \cdot i_3$$
$$u_L = L \cdot \frac{\mathrm{d}}{\mathrm{d}t} i_L$$
$$U_0 = u_1 + u_3$$
$$u_L = u_1 + u_2$$
$$u_3 = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_3$$

ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} i_L = \frac{\sin(t)}{L}$$

Transformation techniques

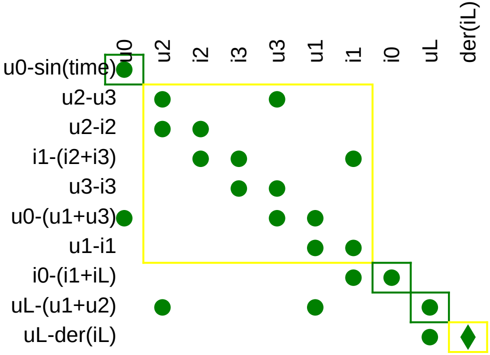
Going from DAE to ODE in general involves many steps:

1. Alias elimination	Get rid of equations and corresponding variables of the form $x \pm y = 0$
2. Index reduction	Perform index reduction until DAE is index 1 (dummy derivatives)
3. Matching	Match variables and equations (Hopcroft-Karp)
4. BLT	Transform the system to block-lower triangular (BLT) form with blocks of minimal size (Tarjan)
5. Tearing	Tear algebraic loops
6. Solve loops	Solve algebraic loops (Newton or LU)

Steps 1 and 5 are optional and done for performance.

BLT example

The block-lower triangular (BLT) transformation is central. Example:



- Allows state derivatives $\dot{x}$ and algebraic variables y to be solved for sequentially (in terms of state x and input u), resulting in ODE
- Non-scalar and/or nonlinear blocks require numerical treatment

Causalization for dynamic optimization

- DAE-constrained optimization traditionally done using full DAE
- Research idea: Utilize some of the transformation techniques for DAE simulation for optimization
- For simulation, goal is to get equivalent ODE
- My goal is instead to get the equivalent (reduced) DAE that is most suitable for numerical optimization

Elimination techniques

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5. Tearing	Tear algebraic loops
~~6. Solve loops~~	Solve algebraic loops (Newton or LU)
7. Sparsity preservation	Undo some parts of steps 4 and 5 to preserve sparsity

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Linear tearing

- Used to improve efficiency when solving $\bar{A}x = b$ when $\bar{A}$ is sparse but without significant structure
- Permute $\bar{A}$ to get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

  such that A is easy to invert and D is small

- Solution is then

$$x_2 = \left(D - CA^{-1}B\right)^{-1} \left(b_2 - CA^{-1}b_1\right)$$
$$x_1 = A^{-1} \left(b_1 - B\left(D - CA^{-1}B\right)^{-1} \left(b_2 - CA^{-1}b_1\right)\right)$$

- $D - CA^{-1}B$ is the Schur complement of block A

Linear tearing in BLT

- In the BLT form, we use linear tearing for linear, non-scalar blocks
- Make A lower triangular (invert by forward substitution)
- Terminology:

  Causalized variables $x_1 \in \mathbb{R}^{n_c}$

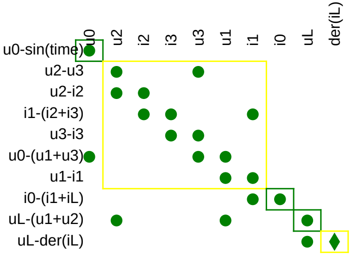
  Tearing variables $x_2 \in \mathbb{R}^{n_t}$

  Causalized equations $Ax_1 + Bx_2 = b_1 \in \mathbb{R}^{n_c}$

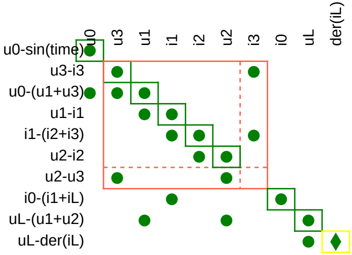
  Tearing residuals $Cx_1 + Dx_2 = b_2 \in \mathbb{R}^{n_t}$
- Inverting $\bar{A}$ has cost $\mathcal{O}((n_c + n_t)^3)$
- Using Schur complement when A is lower triangular, cost is instead $\mathcal{O}(n_c^2 n_t + n_c n_t^2 + n_t^3)$
- Got rid of $\mathcal{O}(n_c^3) \implies$ we want few tearing variables

Tearing example

BLT without tearing



BLT with tearing



Selecting tearing variables and residuals

- Finding a minimal set of tearing variables and residuals such that A is lower triangular is NP-hard
- However, some choices of tearing variables/residuals will cause ill-conditioned Schur complement!
- So even if it were tractable to minimize n_t, would often be bad
- Selection either by heuristic algorithms, or manually by experts
- The automatic tearing in Dymola is a trade secret and one of the major reasons of its success

Nonlinear tearing

- For nonlinear systems $F(x) = 0$, the main idea is the same
- Tear to create partition

$$
\begin{array}{cc}
 & \begin{array}{cc} \quad\quad\quad x_1 \quad\quad\quad & \quad\quad x_2 \quad\quad \end{array} \\
\begin{array}{c} F_1 \\ \\ \\ \\ F_2 \end{array} &
\left[\begin{array}{cccc|ccc}
1 & 0 & \cdots & 0 & * & \cdots & * \\
* & 1 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & 0 & \vdots & \ddots & \vdots \\
* & * & * & 1 & * & \cdots & * \\
\hline
* & * & \cdots & * & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * & * & \cdots & *
\end{array}\right],
\end{array}
$$

where F_1 is lower triangular and constant along diagonal

- F_1 nonlinear w.r.t. x_1, but easy to invert by forward substitution

Skipping algebraic loops

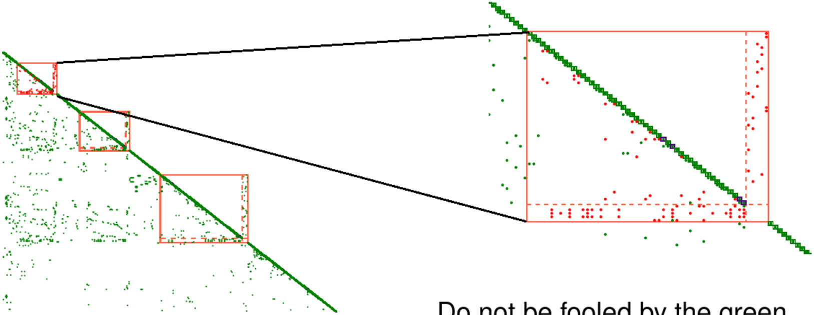
- When transforming all the way to an ODE, we need to numerically solve the tearing residuals
    - LU decomposition for linear blocks, Newton for nonlinear blocks
- The resulting ODE is thus not on closed form, and takes a long time to evaluate the right-hand side of
- Simply leave the tearing residuals, yielding a smaller DAE with cheap residuals
- Consequently no point in solving for the state derivatives $\Longrightarrow$ always choose all state derivatives as tearing variables

Sparsity preservation

- Resulting DAE is much smaller, but usually much denser
- Sometimes the resulting density is crippling for optimization
- Should thus also consider sparsity when tearing
- As far as I know, this is previously unexplored/unpublished territory for dynamic systems (even for simulation)
- Nice ideas from dynamic pivot selection in direct methods for sparse matrices (Markowitz criterion and local minimum fill-in)

A real example

A real BLT form where tearing is important



Do not be fooled by the green landscape! Nonlinearities in the state variables.

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Old Results

Results from 1 year ago without tearing and sparsity preservation

Problem		n_x	n_y	Time [s]
ST-WF	Full	13	27	3.7
	Reduced	13	4	2.0
CCPP	Full	10	123	2.3
	Reduced	10	1	0.9
Dist. Col.	Full	125	1000	12
	Reduced	125	2	94

All of these problems lack algebraic loops $\Longrightarrow$ no use for tearing. But maybe sparsity preservation helps!

New results

Some new problems to demonstrate effects of tearing

Problem		n_x	n_y	Time [s]
Dist. Col.	Full	125	1000	12
	Reduced	125	~~2~~ 63	~~94~~ 4
Fourbar1	Full	2	452	37
	Reduced	2	46	2
HRSG	Full	18	75	8.4
	Reduced	18	14	3.6
Dbl. Pend.	Full	4	124	∞
	Reduced	4	7	3.0

Conclusion

Elimination techniques for dynamic optimization:

- Eliminate algebraic variables by identifying BLT structure and tear non-scalar blocks
- Keep some choice algebraic variables to preserve sparsity
- Employing these techniques reduces solution time by a factor between 2 and ∞
- I have yet to encounter a problem where a suitable combination of these ideas do more harm than good
- No other dynamic optimization software utilizes these techniques

The end

Thank you for listening!

The End