A Radial Basis Function Method for Approximating the Optimal Event-Based Sampling Policy

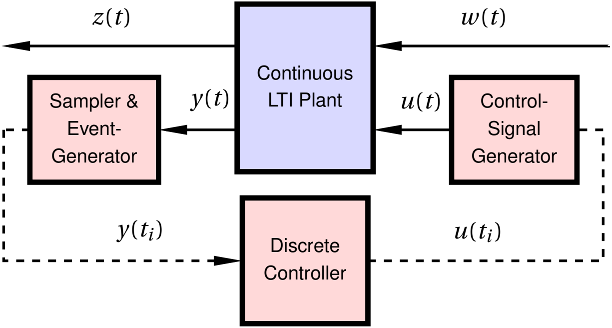
Marcus Thelander Andrén

Friday Seminar

This Talk

- **Part I:** Background on optimal event-based sampling
- **Part II:** Solution using RBF's and linear complementarity
- **Part III:** Some numerical results, and future work

Event-Based LQG Control



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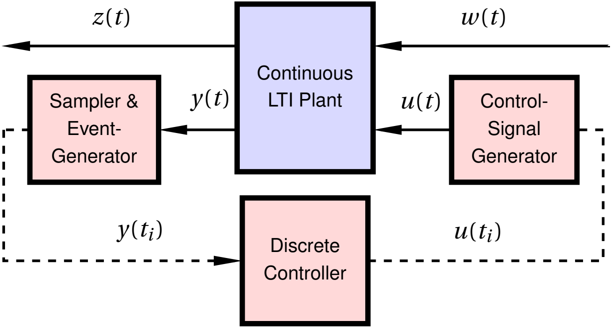

$$\text{Minimize } J_z = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T z(t)^\intercal z(t)\,dt\right] + \rho f$$

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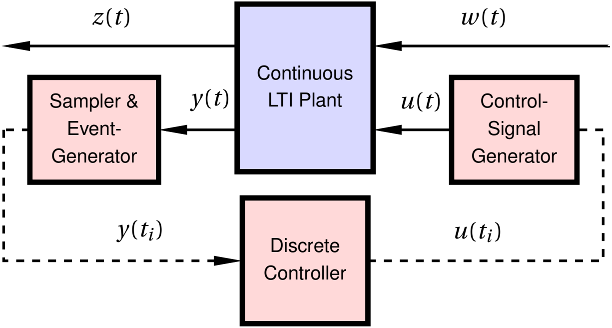

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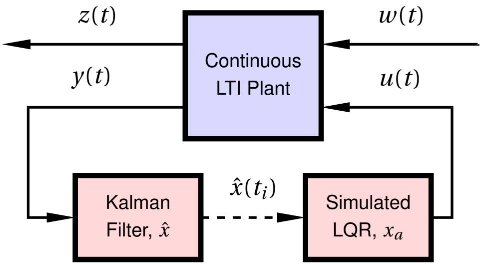
- Controller structure (sampler, discrete controller, signal generator)
- Sampling policy (event-generator)

The Optimal Controller Structure

[Mirkin 2017]: Optimal controller structure can be represented as



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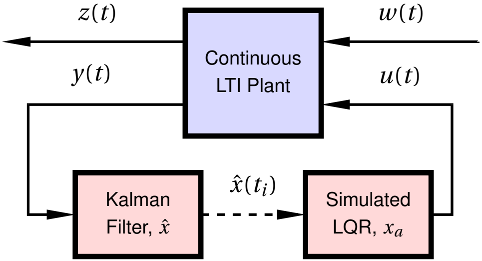

Error: $\tilde{x} \triangleq \hat{x} - x_a,$ **Innovation:** $d\epsilon \sim \mathcal{N}(0, R\,dt),$

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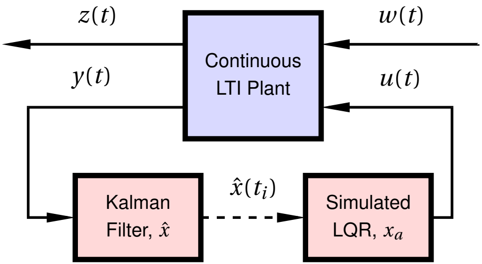

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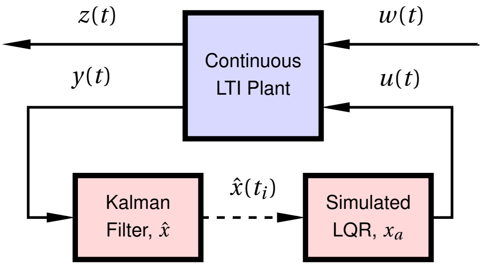

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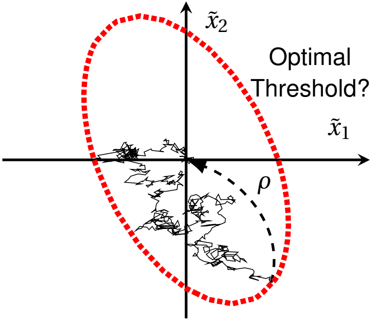
Dynamics: $d\tilde{x} = A\tilde{x}\,dt + d\epsilon$, $\tilde{x}(t_i) = 0$,

Cost: $J_z = \gamma_0 + \underbrace{\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \tilde{x}(t)^\intercal Q \tilde{x}(t)\,dt\right] + \rho f}_{\triangleq J}.$

The Optimal Sampling Policy



The Optimal Sampling Policy

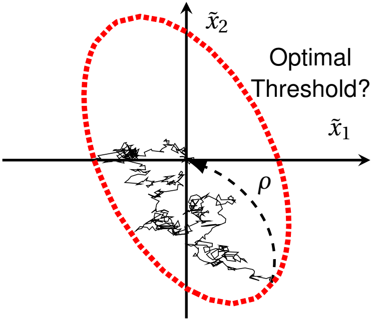
Opt. cost J and value function $V(\tilde{x})$ satisfy [Henningsson 2012]:

Free Boundary PDE:

$$\tilde{x}^\intercal Q\tilde{x} + \tilde{x}^\intercal A\nabla V + \frac{1}{2}\text{Tr}(R\nabla^2 V) \geq J,$$

$$\rho + V(0) - V(\tilde{x}) \geq 0,$$

(with equality in at least one $\forall \tilde{x}$)



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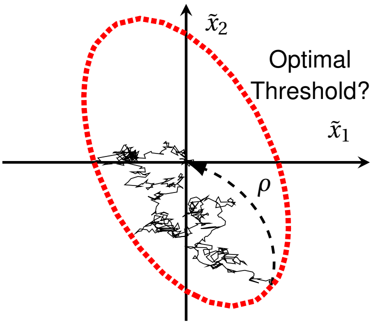
Optimal policy:

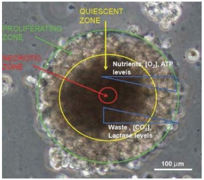
Sample when $\rho + V(0) - V(\tilde{x}) = 0$



Other Examples of Free Boundary Problems

- Heat diffusion with phase-transition ("Stefan problem")

- Biological modeling, e.g. tumor growth and wound healing.



- Valuation of American-style options. (find policy with largest expected payoff)

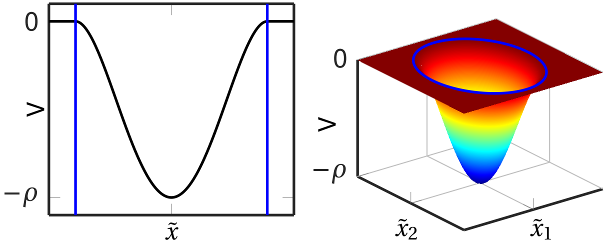
Previous Results

Special case for A=0 [Henningsson 2012]:

$$V(\tilde{x}) = -\frac{1}{4}\max(2\sqrt{\rho} - \tilde{x}^\intercal P\tilde{x}, 0)^2 \qquad (1)$$

where P≻ 0 satisfies: $PRP + \frac{1}{2}\text{Tr}(RP)P = Q$ (2)

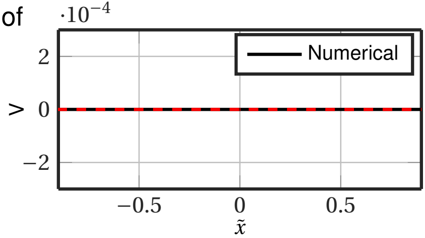
Equation (2) solved by simple scalar search [Andrén et.al 2017]



Previous Results

Finite-difference method for A≠0 [Andrén et.al 2017]:

1. Assume time-dependent version of PDE, and pick an initial V



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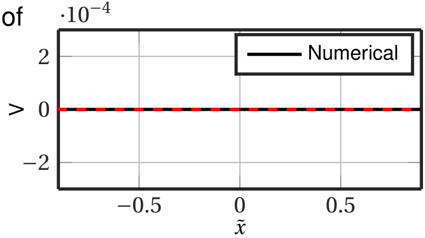

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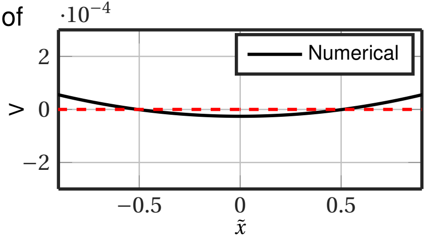

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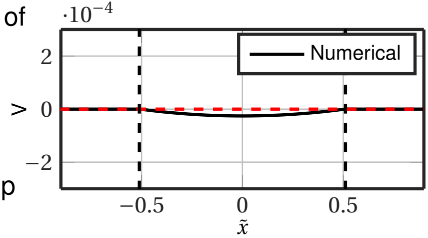

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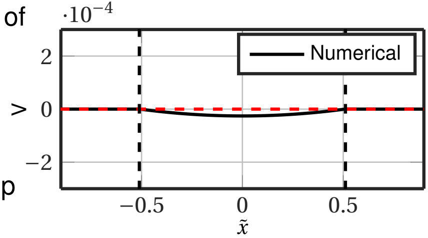

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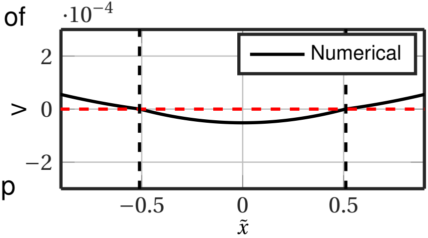

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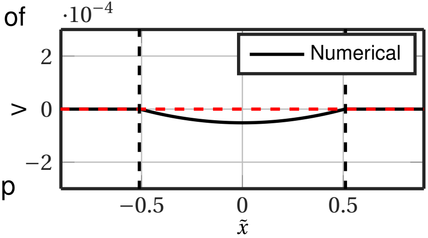

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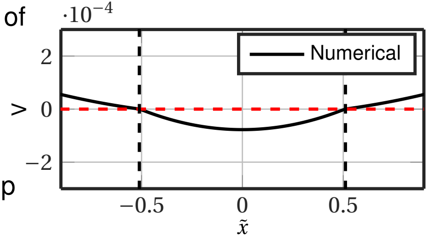

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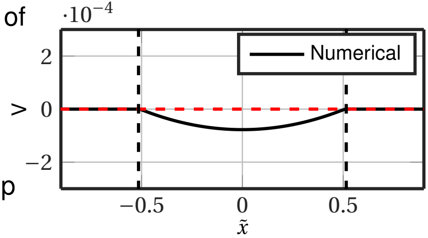

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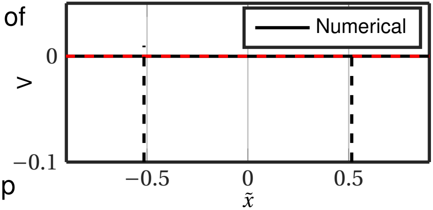

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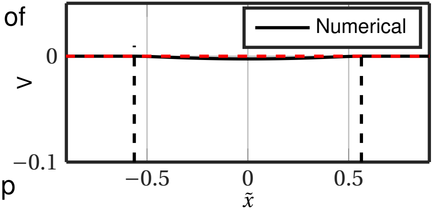

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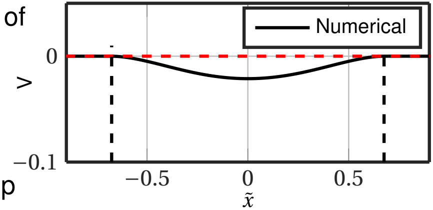

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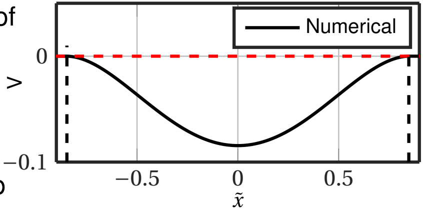

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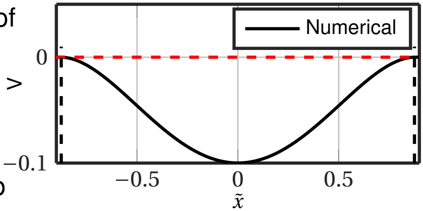

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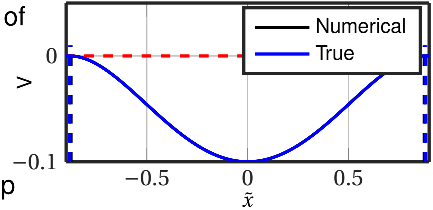


Drawbacks

1. Gridding $\implies$ only suitable for low-dimensional problems
2. Convergence time depends on initial guess of V
3. Time-discretization introduces errors
4. Hard to say when "stationarity" is reached.
5. Solution only available on pre-defined mesh

Part II: Solution using RBF’s and linear complementarity

Radial Basis Functions

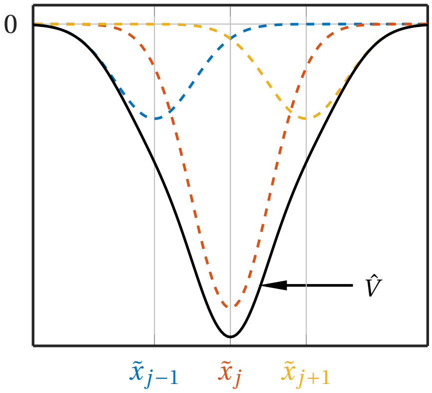
Approximation:

$$\hat{V} = \sum_{j=1}^{n} \alpha_j \phi(\tilde{x} - \tilde{x}_j)$$

Many choices of ϕ exist.

Here we use Gaussian:

$$\phi(\tilde{x} - \tilde{x}_j) = \exp(-c\|\tilde{x} - \tilde{x}_j\|^2)$$



Using the RBF Approximation

1. Set $V(0) = -\rho$, and re-scale system s.t $R = I$.

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The free boundary PDE can be written as:

$$-V \times (\tilde{x}^\top Q \tilde{x} - J + \tilde{x}^\top A^\top \nabla V + \frac{1}{2}\Delta V) = 0, \qquad (3)$$

$$\text{s.t } -V \geq 0, \quad (\tilde{x}^\top Q \tilde{x} - J + \tilde{x}^\top A^\top \nabla V + \frac{1}{2}\Delta V) \geq 0 \qquad (4)$$

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2. Insert $\hat{V} = \sum_j \alpha_j \phi(\tilde{x} - \tilde{x}_j)$

$$-(\sum_j \alpha_j \phi(\tilde{x} - \tilde{x}_j)) \times (\tilde{x}^\intercal Q \tilde{x} - J + \sum_{j=1}^{N} \alpha_j (A\tilde{x} + \frac{\nabla}{2}) \cdot \nabla \phi(x - x_j)) = 0.$$

s.t (4) holds using $\hat{V}$

Using the RBF Approximation

3. Enforce PDE at the collocation points $\{\tilde{x}\}_{j=1}^{n}$:

$$(-\Phi\alpha)^{\top}(\Gamma\alpha+\beta)=0,$$

$$\text{s.t } -\Phi\alpha \geq 0, \quad \Gamma\alpha+\beta \geq 0$$

$$\alpha=\left[\alpha_1,...,\alpha_j,...\alpha_n\right]^{\top}, \quad \beta \in \mathbb{R}^n, \quad \Phi,\Gamma \in \mathbb{R}^{n\times n},$$

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$$\beta_j = \tilde{x}_j^\intercal Q \tilde{x}_j - J,$$
$$\Phi_{i,j} = \phi(\tilde{x}_i - \tilde{x}_j),$$
$$\Gamma_{i,j} = (A\tilde{x}_i + \frac{\nabla}{2}) \cdot \nabla\phi(\tilde{x}_i - \tilde{x}_j)$$

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4. Solve for α!

A Linear Complementarity Problem

Putting $z = -\Phi\alpha$ and $M = -\Gamma\Phi^{-1}$ gives

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Linear Complementarity Problem (LCP):

$$z^\intercal(Mz + \beta) = 0,$$

$$\text{s.t } z \geq 0, \quad Mz + \beta \geq 0.$$

- Proof of existence of unique solution for this particular LCP

A Linear Complementarity Problem

Putting $z = -\Phi\alpha$ and $M = -\Gamma\Phi^{-1}$ gives

Quadratic Program:

$$\min_{z} \; z^\intercal (Mz + \beta),$$

s.t $z \geq 0, \quad Mz + \beta \geq 0.$

- Proof of existence of unique solution for this particular LCP
- Use any solver which handles QP's, e.g `Gurobi`, `OSQP.jl`, `ProximalOperators.jl`...

A Linear Complementarity Problem

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Quadratic Program:

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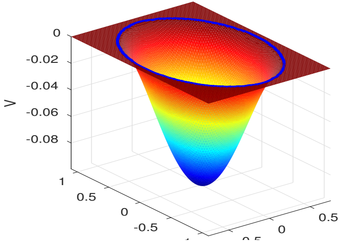
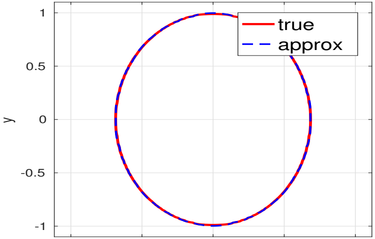
Part III: Some numerical results, and future work

2D, A=0





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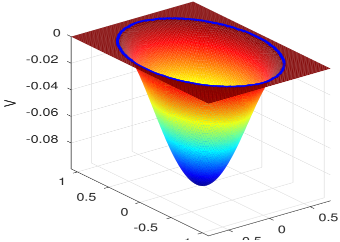
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-0.04
-0.06
-0.08
V
1
0.5
0
-0.5
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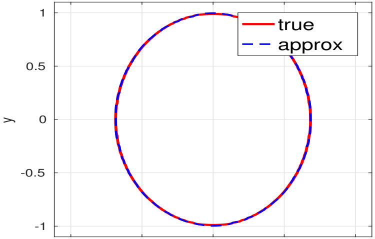
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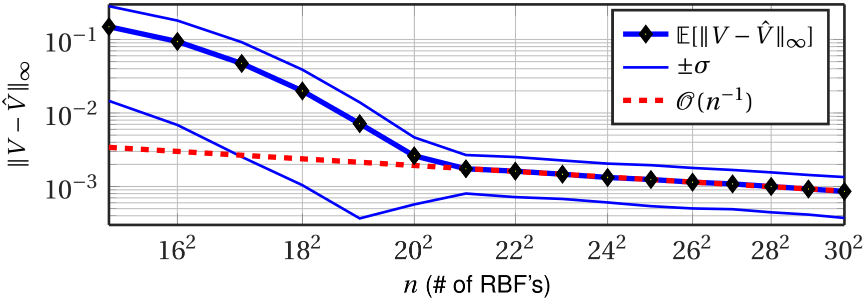
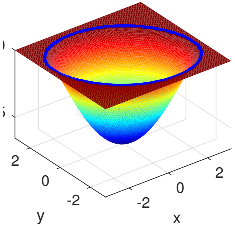
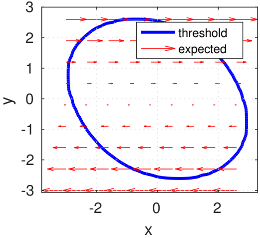

$\mathbb{E}[\|V-\hat{V}\|_\infty]$
$\pm\sigma$
$\mathcal{O}(n^{-1})$
10^{-1}
10^{-2}
10^{-3}
$\|V-\hat{V}\|_\infty$
16^2
18^2
20^2
22^2
24^2
26^2
28^2
30^2
n (# of RBF's)

2D, A≠0

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = R = I,$$





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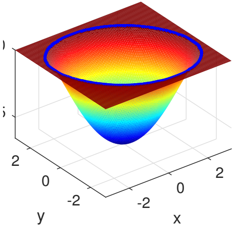
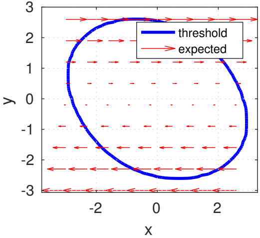
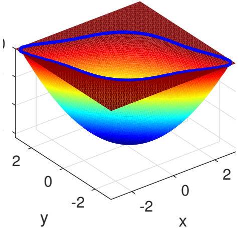
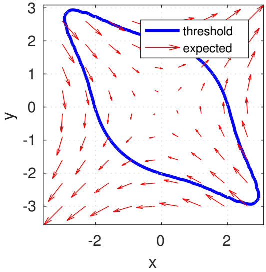


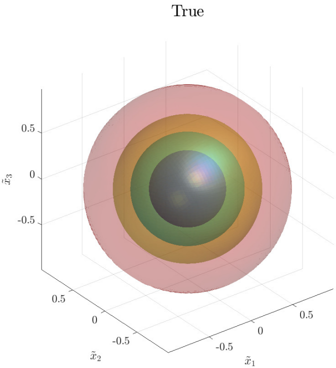


$$A = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \quad Q = R = I,$$



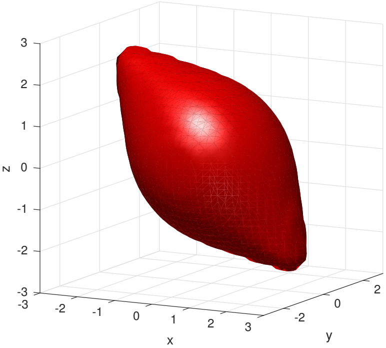


3D, A=0



$$n = 25^3 = 15{,}625, \quad \|V - \hat{V}\|_\infty = 3.1 \times 10^{-4}$$

3D, A≠0



$$A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad R = Q = I, \quad n = 25^3 = 15,625$$

Future Work

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