Nonlinear Control and Servo Systems
Lecture 2

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Today's Goal

To be able to

- *prove local and global stability of an equilibrium point through Lyapunov's method*
- *show stability of a set (for example, a limit cycle) through invariant set theorems*

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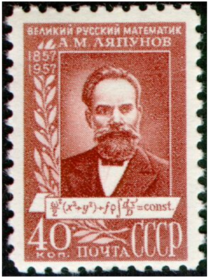
Material

- Slotine and Li: Chapter 3
- Lecture notes

Alexandr Mihailovich Lyapunov (1857–1918)



Master thesis

"On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis

"The general problem of the stability of motion," 1892.

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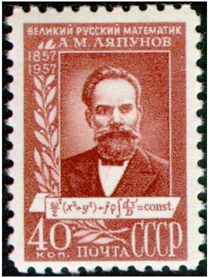

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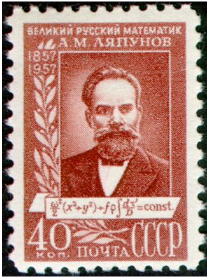

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Lyapunov's idea

If the total energy is dissipated, the system must be stable.

Main benefit

By looking at an energy-like function (a so called Lyapunov function), we might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.

Main question

How to find a Lyapunov function?

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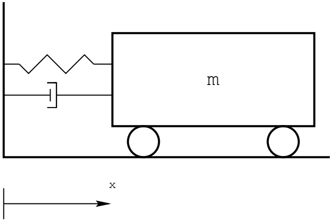
A Motivating Example



$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}, \qquad b, k_0, k_1 > 0$$

The energy can be shown to be

$$V(x, \dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

$$\frac{d}{dt}V(x, \dot{x}) = m\dot{x}\ddot{x} + k_0 x\dot{x} + k_1 x^3 \dot{x} = -b|\dot{x}|^3 < 0, \quad \dot{x} \neq 0$$

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

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globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

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Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\frac{d}{dt} V(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- $\frac{d}{dt} V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

Proof: see p. 62.

A *Lyapunov function* fulfills $V(x_0) = 0$, $V(x) > 0$ for $x \in \Omega$, $x \neq x_0$, and

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{dV}{dx}\dot{x} = \frac{dV}{dx}f(x) \leq 0$$