Nonlinear Control and Servo Systems

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Today's Goal

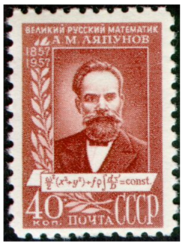
To be able to

- *prove local and global stability of an equilibrium point through Lyapunov's method*
- *show stability of a set (for example, a limit cycle) through invariant set theorems*

Material

- Slotine and Li: Chapter 3
- Lecture notes

Alexandr Mihailovich Lyapunov (1857–1918)



Master's thesis
"On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis
"The general problem of the stability of motion," 1892.

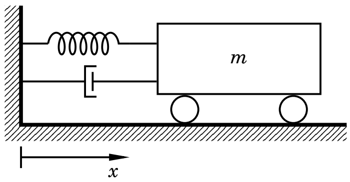
Lyapunov's idea

If the total energy is dissipated, the system must be stable.

Main benefit
By looking at an energy-like function (a so called Lyapunov function), we might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.

Main question
How to find a Lyapunov function?

A Motivating Example



$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} \underbrace{-k_0x - k_1x^3}_{\text{spring}}, \qquad b, k_0, k_1 > 0$$

The energy can be shown to be

$$V(x,\dot{x}) = m\dot{x}^2/2 + k_0x^2/2 + k_1x^4/4 > 0, \qquad V(0,0) = 0$$

$$\frac{d}{dt}V(x,\dot{x}) = m\dot{x}\ddot{x} + k_0x\dot{x} + k_1x^3\dot{x} = -b|\dot{x}|^3 < 0, \qquad \dot{x} \neq 0$$

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem
Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$.
Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\frac{d}{dt}V(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- $\frac{d}{dt}V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

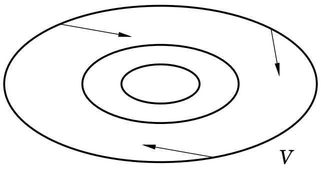
then $x = 0$ is locally asymptotically stable.

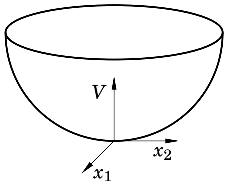
Proof: see p. 62.

Lyapunov Functions (≈ Energy Functions)

A *Lyapunov function* fulfills $V(x_0) = 0$, $V(x) > 0$ for $x \in \Omega$, $x \neq x_0$, and

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{dV}{dx}\dot{x} = \frac{dV}{dx}f(x) \leq 0$$





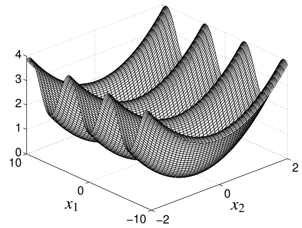
2 min exercise—Pendulum

Show that the origin is locally stable for a mathematical pendulum.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Use as a Lyapunov function candidate

$$V(x) = (1 - \cos x_1)g\ell + \ell^2 x_2^2/2$$



Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $V(x) \to \infty$ as $\|x\| \to \infty$

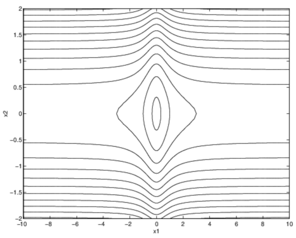
then $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1 + x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



Positive Definite Matrices

A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$.
It is **positive semidefinite** if $x^T M x \geq 0$ for all x.

A symmetric matrix $M = M^T$ is positive definite if and only if its eigenvalues $\lambda_i > 0$. (*semidefinite* ⇔ $\lambda_i \geq 0$)

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

More matrix results

A symmetric matrix $M = M^T$ satisfies the inequalities

$$\lambda_{\min}(M)\|x\|^2 \;\leq\; x^T M x \;\leq\; \lambda_{\max}(M)\|x\|^2$$

(To show it, use the factorization $M = U \Lambda U^*$, where U is a unitary matrix, $\|Ux\| = \|x\|$, U^* is complex conjugate transpose, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.)

For any matrix M one also has

$$\|Mx\| \;\leq\; \lambda_{\max}^{1/2}(M^T M)\|x\|$$

Lyapunov Function for Linear System

Theorem The eigenvalues λ_i of A satisfy $\text{Re}\,\lambda_i < 0$ if and only if: for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$PA + A^T P = -Q$$

Proof of $\underline{\exists Q, P \Rightarrow \text{Re}\,\lambda_i(A) < 0}$: Consider $\dot{x} = Ax$ and the Lyapunov function candidate $V(x) = x^T P x$.

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x = -x^T Q x < 0, \quad \forall x \neq 0$$

$$\Rightarrow \quad \dot{x} = Ax \quad \text{asymptotically stable} \quad \Longleftrightarrow \quad \text{Re}\,\lambda_i < 0$$

Proof of $\underline{\text{Re}\,\lambda_i(A) < 0 \Rightarrow \exists Q, P}$: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$

Lyapunov's Linearization Method

Recall from Lecture 2:

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $x = 0$ is an equilibrium point and that

$$\dot{x} = Ax + g(x)$$

is a linearization.

(1) If $\text{Re}\,\lambda_i(A) < 0$ for all i, then $x = 0$ is locally asymptotically stable.

(2) If there exists i such that $\lambda_i(A) > 0$, then $x = 0$ is unstable.